Vol. 2 – ELECTROMAGNETISM

The Undergraduate Companion to Theoretical Physics

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Contents

The enchanting charms of this sublime science reveal only to those who have the courage to go deeply into it.

— Carl Friedrich Gauss

References

Several textbooks, online courses/resources were referenced heavily (to the extend of making this text completely unoriginal, yet hopefully helpful for revision) throughout the writing of these lecture notes. Using a typical bibliography (research paper style) would be a formidable task.Pinpointing exactly where each reference as been used is quite difficult for such a large and well-referenced subject, and would probably change the writing style to a far too formal one for lecture notes. Therefore we instead list the most relevant below giving a brief comment on which topics they were mostly used for:

- Griffiths *Introduction to Electrodynamics*

  Virtually the only reference needed. Every undergraduate should read this textbook. Period.

- *OU textbooks*

  These textbooks supplemented Griffiths, especially for some niche topics such as supeconductivity. It also discussed EM waves with some more depth.

Mathematical Prelude

Part I

Electrostatics

Fundamentals of Electrostatics

1

1.1 Charge and Charge density

The main question that we will try to answer in electrostatics is: **given the positions and charges of a system of particles, can we evaluate the force exerted on another test charge Q?**

It may seem obvious, after reading this question, what a charge is to the reader, but it is important to define such fundamental quantities.

Definition 1: Electric charge and charge density

Electric charge is a fundamental property of an object, which is responsible for its interaction in electromagnetic fields. Electric charge is quantized, so it only comes in integer multiples of the elementary charge, the charge of the electron:

$$e \approx 1.60217733 \times 10^{-19} \tag{1.1.1}$$

Nevertheless, mathematically we can define a continuous volume charge density $\rho(\mathbf{r})$ such that the charge enclosed in a volume $\mathcal{V}$ is:

$$Q = \int_V d^3\mathbf{r}\ \rho(\mathbf{r}) \tag{1.1.2}$$

An important application of this definition is the charge density of a system of N point charges:

$$\rho(\mathbf{r}) = \sum_{k=1}^{N} q_k \delta(\mathbf{r} - \mathbf{r}_k) \tag{1.1.3}$$

We also enunciate the following properties of matter (in addition to the quantization of charge):

(i) Charge is scalar, and can be described by one real number with an associated unit of measurement.

(ii) Charge is additive, so the sum of the charges in a region is equal to the charge within the given region.

(iii) Charge is conserved locally: the total charge in any region of space remains constant unless there is some charge flowing through its boundary. [1]

(iv) Charge is invariant under Lorentz transformations

1.2 Electromagnetic interactions

The electromagnetic force is one of the **four fundamental interactions** (the other three being gravitational, strong and weak nuclear forces).

The electromagnetic force decreases with increasing separation. Indeed, it is about 10^{36} times stronger than the gravitational force.

Two more comparable forces are the nuclear forces, which for extremely short separations, beat the electromagnetic force. For separations greater than 10^{-12} m however, the latter becomes more prevalent.

This means that the fundamental interactions in chemistry is the electromagnetic force.

The electromagnetic force also depends on the velocities of the charges involved, as can be seen below:

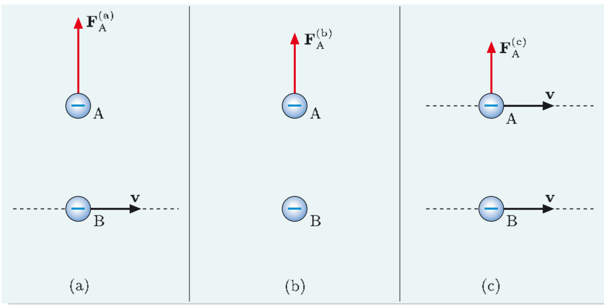


Figure 1.1. Interaction between point particles moving at different relative speeds

Here, $|\mathbf{F}_A^{(a)}| > |\mathbf{F}_A^{(b)}| > |\mathbf{F}_A^{(c)}|$.

We can use these results to explain why two steady currents flowing through a wire in the same direction are attracted to each other. Indeed, we can consider the two wires as composed of positively charged ions staying still and negatively charged electrons in motion. The electric attraction is therefore null, since both wires are neutral [2]. However, the motion of the electrons reduces the repulsive force between them as is shown in Figure 1.1(c). This reduction in the repulsive force is not balanced by a reduction in the attractive forces, so that the overall effect is an attraction between the currents.

[1] what about pair production? If from a neutral boson we create a pair of electron and positron. Then charge is still conserved locally, because if we concentrate on the region isolated on the boson which then splits into the electron and positron, charge is conserved. If we instead concentrate on just the electron produced, then the positron must have moved through the region at some point or another. Hence charge is still conserved locally, since any difference in charge is compensated by the current through the region.

[2] the electrons are attracted and repelled by the positive ions and negative electrons respectively.

This force is the magnetic force, which acts on electric currents. So we see that the electromagnetic force on a stationary point charge is electric in nature, whereas the electromagnetic force on a moving particle is both magnetic and electric.

In summary, electric forces do not depend on the motion of the charges that feel them, but on the motion of the charges that exert them. The magnetic forces instead depend on both.

1.3 Coulomb's Law

We now present the cornerstone of electrostatics, this is perhaps the only physical law (combined with superposition as we shall see) that is required to formulate all the laws of this part.

Coulomb's Law

The force acting on a charge q_1 with position $\mathbf{r}_1$ due to a charge q_2 with position $\mathbf{r}_2$ is given by:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) \tag{1.3.1}$$

where $\varepsilon_0 \approx 8.85 \times 10^-12 \mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}$ is the permittivity of free space.

Coulomb's law alone does not help much, since it only allows us to calculate the force due to a single point charge, but what if there are several particles? Luckily, the superposition principle, which states that the interaction between two particles is unaffected by the presence of other particles, simplifies this problem. Indeed, for a given distribution of particles it suffices to calculate the force due to each and them sum them together. An immediate consequence of Coulomb's law combined with the superposition principle is that:

The Superposition principle and Electric field

The force that a charge distribution $\rho(\mathbf{r}')$ exerts on $\rho^*(\mathbf{r})$ is given by:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\int\int \rho^*(\mathbf{r})\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}'d^3\mathbf{r} \tag{1.3.2}$$

We can then define the electric field due to the charge distribution $\rho(\mathbf{r}')$, which is the force per unit charge, to be:

$$\mathbf{E} = \frac{\mathbf{F}}{\int d^3\mathbf{r}\rho^*(\mathbf{r})} = \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{r}'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1.3.3}$$

It immediately follows that for a discrete system of particles of charge q_j and position

$\mathbf{r}_j$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_j}{|\mathbf{r} - \mathbf{r}_j|^3} (\mathbf{r} - \mathbf{r}_j) \tag{1.3.4}$$

There are important consequences of this definition and result:

(i) The electric force follows an inverse square law. As the distance increases by a certain factor, the force decreases by that factor squared.

(ii) The force between two like charges is repulsive, since it acts along $\mathbf{r} - \mathbf{r}'$,away from the direction of the source. Similarly, the force between opposite charges is attractive, it acts along $\mathbf{r}' - \mathbf{r}$ in the direction of the source.

Empirical evidence for Coulomb's law

It may seem to the reader as if Coulomb's law is just a fundamental fact of nature that we must just accept without evidence. However, if unmotivated, this view is very nonphysical. Indeed, Coulomb's law was not just some equation derived mathematically, but was instead empirically found through experiments. Below is a table showing the uncertainty in the value of $n = 2$, where we assume that the electrostatic force decreases as $\frac{1}{r^n}$.

Date	Physicist	Δn
1769	Robinson	± 0.06
1773	Cavendish	± 0.03
1785	Coulomb	± 0.1
1873	Maxwell	$\pm 10^{-5}$
1936	Plimpton	$\pm 10^{-9}$
1970	Bartlett	$\pm 10^{-13}$
1971	Williams	$\pm 10^{-16}$

Testing of Coulomb's law also was performed at both very large and very short distances. Rutherford's famous gold foil experiment showed that Coulomb's law was applicable at atomic length scales. At large distances, one must resort to quantum field theory. Provided that the photon has no mass, QFT predicts Coulomb's inverse square law. In the case in which the photon was not massless, then one would instead find:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}^3} \left(1 + \frac{r_{12}}{a}\right) e^{-\frac{r_{12}}{a}} \mathbf{r}_{12} \tag{1.3.5}$$

where $a = \frac{\hbar}{mc}$ and $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$.

When can Coulomb's law be used

Coulomb's law can be used provided that both:

(i) the source charges are at rest (this will be discussed at the end of the course) [3]

(ii) the locations of all the relevant charges are known

1.4 Symmetry arguments part 1

Although the formulation of the laws of electrostatics may seem analytical in nature, they can be interpreted geometrically through the use of **field lines**.

Definition 2: ELECTRIC FIELD LINES

Electric Field lines are a collection of continuous lines that are tangential at every point in space to the electric vector field, and that show the direction in which a positive test charge would move if placed on the line.

(i) They radiate from positive charges to negative charges,
(ii) the density of field lines represents the strength of the electric field
(iii) if two field lines cross, then they must have the same value of zero (otherwise the particle would have to move in two different directions)
(iv) they inherit the symmetry of the source charges

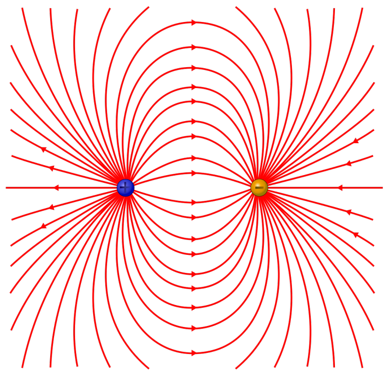


Figure 1.2. Electric field lines for an **electric dipole**, an arrangement of two opposite charges

Symmetry arguments are an important tool in electromagnetism that allow us to simplify the vector calculus required in the calculations.

Suppose a charge distribution benefits from some sort of symmetry, for example rotation. If the distribution is rotated appropriately, so that it is indistinguishable from the original orientation, then the electric field must also be indistinguishable, since there is no preferred orientation in space.

[3] luckily, the corrections to Coulomb's law if the charges move are in the order of $\frac{v^2}{2c^2}$, so for non-relativistic particles Coulomb's law still holds approximately.

Symmetry arguments

If a charge distribution follows a certain symmetry that leaves it unchanged, then the electric field produced will also respect the same symmetry.

Take for example a charge distribution that is **spherically symmetric**, so that any rotation about any axis through its center will leave it unchanged. By symmetry, the electric field must also be radial. If it had any other component (azimuthal for example), then the rotation would effect the electric field.

Another form of symmetry is **cylindrical symmetry**, where any rotation about a fixed axis of symmetry leaves the object unchanged. Infinite cylinders also have translational symmetry along the axis because if we move it along its axis of symmetry it is left unchanged. This suggests that the electric field due to an infinite cylinder is radial and the same in magnitude for all points that are equidistant from the axis of symmetry.

One final form of symmetry, that is much deeper than the ones we have discussed, is **time reversal symmetry**, which asserts that electromagnetic interactions are unchanged if we reverse the flow of time. With these discussions set, let us look at some examples of how to find the electric field of simple geometrical arrangements of charges.

Thunderstorms maintain the Earth's electric balance

Although many may have heard of the Earth's magnetic field, our planet also hosts its own electric field of magnitude of about 100N/C. Indeed, the surface of the earth carries a charge of about -5×10^5C, whereas the upper atmosphere carries a positive charge. This produces a current small current transporting positive charges to the surface, and which tries to neutralize the Earth's electric field. However, thunderstorms save the day by transmitting an upwards current (in reality it is the negative charge moving downwards, but this is equivalent to an upwards current of positive charges).

The way thunderstorms occur can be explained through electrostatics. Indeed, the electric field just below a thunderstorm points upwards, and is several times stronger than normal. When the electric field reaches the breakdown field of air, the atmosphere get ionized, and the negative charges on the bottom of the thunderstorm find a path to strike the earth's surface. Overall, these 40 000 daily thunderstorms maintain a relatively constant electric field.

Figure 1.3. A thunderstorm

1.5 Divergence and Curl of E

Divergence

Let us now consider the divergence of the electric field. From Coulomb's law we find that, setting $|\mathbf{r}-\mathbf{r}'|^3 = \mathbf{r}-\mathbf{r}'$:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \nabla \cdot \left(\frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{r}') \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}' \right) \tag{1.5.1}$$

$$= \frac{1}{4\pi\varepsilon_0} \int \nabla \cdot \left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \right) \rho(\mathbf{r}') d^3\mathbf{r}' \tag{1.5.2}$$

$$= \frac{1}{4\pi\varepsilon_0} \int \delta^3(\mathbf{r}-\mathbf{r}') \rho(\mathbf{r}') d^3\mathbf{r}' \tag{1.5.3}$$

$$= \frac{1}{4\pi\varepsilon_0} (4\pi\rho(\mathbf{r})) = \frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{1.5.4}$$

We therefore have found **Gauss' Law** in differential form (taking the triple integral and using the divergence theorem we also find the integral form):

Gauss' Law in Integral and Differential form

The electric field due to some charge distribution $\rho(\mathbf{r})$ satisfies:

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0} \iff \oiint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\varepsilon_0} \tag{1.5.5}$$

where $\mathcal{S}$ is some closed surface and Q_{enc} is the charge enclosed by it.

Gauss' Law is especially useful when dealing with the symmetries mentioned previously:

(i) Spherical symmetry: use a spherical surface as $\mathcal{S}S$, with $d\mathbf{a} = r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}$.

(ii) Cylindrical symmetry: use a cylindrical surface as $\mathcal{S}S$ with $d\mathbf{a} = sdzd\theta\hat{\mathbf{s}}$

(iii) Plane symmetry: use a **Gaussian pillbox**, a rectangular prism of infinitesimal width, with $d\mathbf{a} = dxdy\hat{\mathbf{z}}$.

It turns out that Gauss' law doesn't hold only for electrostatic problems, but is a fundamental law of electromagnetism, and applies for electrodynamics situations as well.

Furthermore, the differential form of Gauss' law provides a local description of the charge density at a given point using the divergence of the electric field at that point. Any knowledge of other distant charges is required.

Curl

Let us now consider the curl of $\mathbf{E}$

$$\nabla \times \mathbf{E}(\mathbf{r}) = \nabla \times \left(\frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' \right) \tag{1.5.6}$$

$$= \frac{1}{4\pi\varepsilon_0} \int \nabla \times \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \rho(\mathbf{r}') d^3\mathbf{r}' = 0 \tag{1.5.7}$$

Therefore:

Curl of E

The electrostatic field $\mathbf{E}$ is **conservative/irrotational**, so that:

$$\nabla \times \mathbf{E} = 0 \iff \oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.5.8}$$

around any closed path $\mathcal{C}$. This also means that the line integral:

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} \tag{1.5.9}$$

is path independent, and is well defined given the endpoints $\mathbf{a}, \mathbf{b}$.

1.6 The Electric Potential

The fact that the electric field is conservative allows us to define an electric potential:

Definition 3: Electric potential

We define the **electric potential** with respect to some reference $\mathcal{O}$ at which the electric field vanishes:

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \tag{1.6.1}$$

It can be interpreted as the potential energy of a unit charge placed at $\mathbf{r}$ in a field $\mathbf{E}$. It follows from the Fundamental Theorem of Line Integrals that:

$$V(\mathbf{r}) = V(\mathbf{r}) - V(\mathcal{O}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}}^{\mathbf{r}} (\nabla V) \cdot d\mathbf{l} \implies \mathbf{E} = -\nabla V \tag{1.6.2}$$

The potential difference between two points $\mathbf{a}$ and $\mathbf{b}$ is reference independent:

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} \tag{1.6.3}$$

Notice that dealing with the electric potential is much easier than the electric field, since it is a scalar quantity. Therefore, there is no need to work with unit vectors and directions.

Note that the superposition principle of the electric field implies the superposition principle for the electric potential:

$$V(\mathbf{r}) = \sum_i V_i(\mathbf{r}) \tag{1.6.4}$$

This allows us to find the potential due to a localized charge distribution, given the potential due to a single charge. More specifically, for a point charge at the origin, we can set the reference $\mathcal{O}$ to be at infinity, and the path to be radial in the direction $\mathcal{O} \longrightarrow \mathbf{r}$. Then:

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \cdot d\mathbf{r} = -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2} dr = \frac{q}{4\pi\varepsilon_0 r} \tag{1.6.5}$$

so that by the principle of superposition:

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau} \tag{1.6.6}$$

Potential due to charge distribution

The potential at $\mathbf{r}$ due to a charge distribution $\rho(\mathbf{r})$ is given by:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau \tag{1.6.7}$$

1.7 Work and Energy

Suppose we have a stationary configuration of source charges, producing an electrostatic field $\mathbf{E}$. Now we take some test charge Q, and try to move from point $\mathbf{a}$ to point $\mathbf{b}$. What is the work that must be done on the test charge for this to occur?

We know that the force on the test charge is:

$$\mathbf{F} = Q\mathbf{E} \tag{1.7.1}$$

Since $\Delta \times \mathbf{E} = 0$, this means that it does not matter what path we take the particle through, only the endpoints so:

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q(V(\mathbf{b}) - V(\mathbf{a})) \tag{1.7.2}$$

This means that electrostatic configurations have some sort of electric potential energy, due to the work that is required to maintain the configuration static and not have particles flying out in all directions.

This potential energy, by energy conservation, must therefore be equal to the work done assembling the configuration.

Consider for example a discrete configuration of n charges q_i with position vectors $\mathbf{r}_i$.

The work required to place the first particle is zero, since there is no electric field to begin with initially.

Next, work must be done to move the second particle into place, since it is moving in the presence of the electric field due to the first particle. This work is equal to:

$$W_2 = q_2 V_1(\mathbf{r}_2) = \frac{q_2}{4\pi\varepsilon_0}\left(\frac{q_1}{|\mathbf{r}_2 - \mathbf{r}_1|}\right) \tag{1.7.3}$$

Similarly, the work required to move the third charge q_3 in place at position $\mathbf{r}_3$ is:

$$W_3 = q_3(V_1(\mathbf{r}_3) + V_2(\mathbf{r}_3)) = \frac{q_3}{4\pi\varepsilon_0}\left(\frac{q_1}{|\mathbf{r}_3 - \mathbf{r}_1|} + \frac{q_2}{|\mathbf{r}_3 - \mathbf{r}_2|}\right) \tag{1.7.4}$$

We then see that the general formula for the work done in moving the ith charge is:

$$W_i = q_i \sum_{j<i} V_j(\mathbf{r}_i) = \frac{q_j}{4\pi\varepsilon_0} \sum_{j<i} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{1.7.5}$$

where the $j < i$ avoids counting the interaction between two particles twice.

If we perform this sum for all n charges then we find that:

$$\boxed{W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}} \tag{1.7.6}$$

Alternatively, we could count each interaction twice, and smply divide by two to find that:

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{1.7.7}$$

This treatment has however assumed that the charge configuration is discrete. In the case of a continuous charge distribution ρ, the work required to assemble it is given by:

$$W = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})V(\mathbf{r})d^3\mathbf{r} \tag{1.7.8}$$

where $\mathcal{V}$ is the region occupied by the distribution.

We can now use Gauss' law to write that: $\rho = \varepsilon_0 \nabla\cdot\mathbf{E}$ and thus:

$$\begin{aligned} W &= \frac{\varepsilon_0}{2}\int_{\mathcal{V}} \nabla\cdot\mathbf{E}V(\mathbf{r})d^3\mathbf{r} && (1.7.9)\\ &= \frac{\varepsilon_0}{2}\left(-\int_{\mathcal{V}} \mathbf{E}\cdot\nabla V\ d^3\mathbf{r} + \oint_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{a}\right) && (1.7.10)\\ &= \frac{\varepsilon_0}{2}\left(\int_{\mathcal{V}} E^2\ d^3\mathbf{r} + \oint_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{a}\right) && (1.7.11) \end{aligned}$$

We can allow $\mathcal{V}\to\mathbb{R}^3$, by defining ρ to be null outside the region occupied by the charge configuration, then the surface integral must vanish since it encloses all charges. Hence we find that:

$$\boxed{W = \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3} E^2\ d^3\mathbf{r}} \tag{1.7.12}$$

Potential energy of a charge distribution

The potential energy of a discrete distribution of n charges with position vectors $\mathbf{r}_i$ is:

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{1.7.13}$$

For a continuous charge distribution, instead, the potential energy is given by:

$$W = \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3} E^2\ d^3\mathbf{r} = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})V(\mathbf{r})d^3\mathbf{r} \tag{1.7.14}$$

1.8 Boundary conditions

1.9 Conductors and Capacitance

Conductors and BVPs

2

2.1 Poisson's equation

The main goal of this chapter will be to develop an array of mathematical tools that can be employed to solve Poisson's equation. Often, when the charge density is specified over some region in space, one cannot solve Coulomb's law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}\text{'}(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau' \tag{2.1.1}$$

in quadratures, since integrals can be very tricky. In such cases, one can try instead to solve Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{2.1.2}$$

Uniqueness theorem

Consider **Poisson's equation** $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$, where $\rho(\mathbf{r})$ is specified on a region $\mathcal{V}$ with boundary $\mathcal{S}$. We define the boundary conditions as follows:

(i) Neumann condition: $\hat{\mathbf{n}}\cdot\nabla V = \frac{\partial V}{\partial \mathbf{n}} = g(\mathbf{r})$ on $\mathcal{S}$, with outward normal $\hat{\mathbf{n}}$.

(ii) Dirichlet condition: $V(\mathbf{r}) = f(\mathbf{r})$ on $\mathcal{S}$

for some f, g. Then either:

(i) $V(\mathbf{r})$ is unique or,

(ii) $V(\mathbf{r})$ is unique up to a constant

Proof. Suppose we have two solutions, V_1 and V_2, to Poisson's equation satisfying one of the above boundary conditions. By the linearity of Poisson's equation, $V = V_2 - V_1$ must satisfy:

$$\nabla^2 V = 0 \tag{2.1.3}$$

called **Laplace's equation**. Moreover, it satisfies the boundary conditions:

(i) $\frac{\partial V}{\partial \mathbf{n}} = g(\mathbf{r}) - g(\mathbf{r}) = 0$ on $\mathcal{S}S$, with outward normal $\hat{\mathbf{n}}$.

(ii) $V(\mathbf{r}) = f(\mathbf{r}) - f(\mathbf{r}) = 0$ on $\mathcal{S}$

These two can be combined into one condition, $V\frac{\partial V}{\partial \mathbf{n}} = 0$ on $\mathcal{S}$. Then, using the divergence theorem:

$$\int_{\mathcal{V}} \nabla \times (V\nabla V) d\tau = \int_{\mathcal{S}} (V\nabla V) \cdot d\mathbf{a} = 0 \tag{2.1.4}$$

$$\iff \int (\nabla V)^2 + V\cancelto{0}{\nabla^2 V} d\tau = 0 \tag{2.1.5}$$

$$\iff \int (\nabla V)^2 d\tau = 0 \tag{2.1.6}$$

Since the potential is real, $(\nabla V)^2 \geq 0$, so we need $\nabla V = 0$ everywhere in $\mathcal{V}$. Consequently, for any constant c:

$$V = c \implies V_1 = V_2 + c \tag{2.1.7}$$

Applying the boundary conditions:

(i) $V_1 = V_2 + c$

(ii) $V = 0 \implies V_1 = V_2$.

as desired. □

What is the physical interpretation of the Neumann and Dirichlet conditions? Recall from the previous chapter the boundary conditions for the electric field across a surface $\mathcal{S}$ carrying charge density σ:

$$\left.\left(\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n}\right)\right|_{\mathcal{S}} = -\frac{\sigma_f}{\varepsilon_0} \tag{2.1.8}$$

which is the Neumann condition. The Dirichlet condition instead simply specifies the potential over the surface of some conductor.

Hence, the uniqueness theorem implies that:

The potential in a volume $\mathcal{V}$ containing conductors is uniquely determined if the charge on the conductors or the potential on the conductors is specified.

This is a very useful result. If by some magical means we are able to solve Poisson's or Laplace's equation for some conductors and boundary conditions, then we know it is the only possible solution.

2.2 Method of image charges

One of these "magical means" is the method of image charges. This method is best illustrated with an example.

Charge q over grounded conducting plane

Consider a point charge q held at a distance $z = h$ from a grounded conducting plate in the xy-plane. What is the potential above the plane?

One might naively say that the potential is simply that of a point charge q. However, it is important to note that there will be induced charges on the conductor, so as to make the electric field inside the conductor zero. These induced charges will themselves contribute the potential. But to find the induced charges using boundary conditions, we need the potential! We find ourselves in an infinite loop, clearly the tools we have developed until now are not enough.

Let us forget for now the conductor, and place instead another charge $-q$ at $z = -\hbar$. The potential V' of this configuration is.

$$V'(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2 + y^2 + (z-h)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+h)^2}}\right) \tag{2.2.1}$$

The original problem boiled down to solving $\nabla^2 V = \frac{q\delta(\mathbf{r}-\mathbf{r}_0)}{\varepsilon_0}$, where $\mathbf{r}_0 = h\hat{\mathbf{z}}$ with boundary condition:

(i) $V(z = 0) = 0$ since the conductor is grounded

(ii) $V \to 0$ as $r \to \infty$

Interestingly, our two-charge problems satisfies:

$$\nabla^2 V'(x, y, z) = \frac{q}{4\pi\varepsilon_0}\left(\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}_0|} - \nabla^2 \frac{1}{|\mathbf{r}+\mathbf{r}_0|}\right) \tag{2.2.2}$$

$$= -\frac{q\delta(\mathbf{r}-\mathbf{r}_0) - \delta(\mathbf{r}+\mathbf{r}_0)}{\varepsilon_0} \tag{2.2.3}$$

Now, since we are interested in the potential above the plane, we may assume $z \geq 0$ so that:

$$\nabla^2 V'(x, y, z) = -\frac{q\delta(\mathbf{r}-\mathbf{r}_0)}{\varepsilon_0} \tag{2.2.4}$$

which is precisely Poisson's equation for the charge-conductor problem. Let's see if the boundary conditions are also satisfied:

(i) when $z = 0$, $V' = 0$.

(ii) when $r \to \infty$, $V' \to 0$

Hence, V' must be a solution to the conductor-charge problem, and by the uniqueness theorem it can be the **only** solution. Thus:

$$\boxed{V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2 + y^2 + (z-h)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+h)^2}}\right), \; z \geq 0} \tag{2.2.5}$$

Using this expression for the potential, and the boundary conditions for electric fields, we can calculate the charge induced on the surface of the plane:

$$\sigma_{ind}(x,y) = -\varepsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} \tag{2.2.6}$$

$$= -\frac{q}{4\pi}\left(\frac{-(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{z+d}{(x^2+y^2+(z-d)^2)^{3/2}}\right) \tag{2.2.7}$$

$$= \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}} \tag{2.2.8}$$

Thus, the total charge induced on the plane is:

$$Q_{ind} = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r dr d\phi = -q \tag{2.2.9}$$

as expected.

Grounded conducting spherical shell containing charge q

We can use the method of image charges for other symmetrical configurations as well. Consider the problem of determining the potential outside a grounded conducting sphere of radius R, containing a point charge q a distance a from its center. Thus we need to solve Poisson's equation with boundary conditions:

(i) $V(R) = 0$

(ii) $V \to 0$ as $r \to \infty$

We consider an entirely different problem, which will hopefully solve Poisson's equation and the above boundary conditions. We replace the spherical conductor by a charge $q' = kq$ a distance b from the center of the sphere.

The potential of this configuration is:

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{q}{r_+} + \frac{kq}{r_-}\right) \tag{2.2.10}$$

where:

$$r_+ = \sqrt{r^2+a^2-2ra\cos\theta} \tag{2.2.11}$$

$$r_- = \sqrt{r^2+b^2-2rb\cos\theta} \tag{2.2.12}$$

Imposing the first boundary condition with $r = R$:

$$\frac{1}{\sqrt{R^2+a^2-2Ra\cos\theta}} + \frac{k}{\sqrt{R^2+b^2-2Rb\cos\theta}} = 0 \tag{2.2.13}$$

$$\iff \sqrt{R^2+b^2-2Rb\cos\theta} = -k\sqrt{R^2+a^2-2Ra\cos\theta} \tag{2.2.14}$$

$$\iff R^2+b^2-2Rb\cos\theta = k^2(R^2+a^2-2Ra\cos\theta) \tag{2.2.15}$$

We can equate the terms of $\cos\theta$ to find that $k^2 = \frac{b}{a}$. Substituting this:

$$R^2 + b^2 = \frac{b}{a}R^2 + ab \tag{2.2.16}$$

$$\iff \left(1 - \frac{b}{a}\right)R^2 = \left(1 - \frac{b}{a}\right)ab \tag{2.2.17}$$

$$\implies b = \frac{R^2}{a},\ k = -\frac{R}{a} \tag{2.2.18}$$

since the image charge is opposite in sign to the real charge [1]. Thus, we find that the image charge $q' = -\frac{R}{a}q$ must be placed at a distance $b = \frac{R^2}{a}$ from the center of the spherical shell. The potential for this configuration is:

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{R/a}{\sqrt{r^2 + \frac{R^4}{a^2} - 2\frac{rR^2}{a}\cos\theta}}\right) \tag{2.2.19}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{R}{\sqrt{R^4 + a^2r^2 - 2arR^2\cos\theta}}\right) \tag{2.2.20}$$

$$\boxed{V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{R^2 + \frac{a^2r^2}{R^2} - 2ar\cos\theta}}\right),\ r \geq R} \tag{2.2.21}$$

The induced surface charge density σ_{ind} is then given by:

$$\sigma_{ind}(\theta) = -\varepsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = \frac{q}{4\pi R}\frac{R^2 - a^2}{(R^2 + a^2 - 2arR\cos\theta)^{3/2}} \tag{2.2.22}$$

so that the total induced charge is:

$$Q_{ind} = \frac{q}{4\pi R}\int_0^\pi \frac{R^2 - a^2}{(R^2 + a^2 - 2arR\cos\theta)^{3/2}}R^2\sin\theta d\theta \tag{2.2.23}$$

$$= \frac{qR(R^2 - a^2)}{2}\int_0^\pi \frac{\sin\theta d\theta}{(R^2 + a^2 - 2arR\cos\theta)^{3/2}} \tag{2.2.24}$$

$$= \frac{qR(R^2 - a^2)}{2}\left[-\frac{1}{2aR}\frac{2}{\sqrt{R^2 + a^2 - 2aR\cos\theta}}\right]_0^\pi \tag{2.2.25}$$

$$= -\frac{qR(R^2 - a^2}{2aR}\left(\frac{1}{R+a} + \frac{1}{R-a}\right) \tag{2.2.26}$$

$$= -\frac{qR}{a} = -q \tag{2.2.27}$$

since $a \geq R$, as expected.

So, in general, to use the method of image charges, we need to replace the conductor with an(several) image charge(s) such that the resulting potential solves the same Poisson's equation and satisfies the same boundary conditions.

[1] one can also check manually that only the negative solution satisfies (2.2.14)

2.3 Solution methods in cartesian coordinates

Consider Laplace's equation in cartesian coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0} \tag{2.3.1}$$

2.4 Solution methods in cylindrical coordinates

Consider Laplace's equation for a cylindrically symmetric potential $V(r, \phi)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \tag{2.4.1}$$

Again, we can use separation of variables. Let's assume the solution is of the form $V(r) = R(r)\Phi(\phi)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\Phi(\phi)\frac{dR}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Phi}{d\phi^2} = 0 \tag{2.4.2}$$

$$\iff \frac{1}{r}\Phi(\phi)\left(\frac{dR}{dr} + r\frac{d^2R}{dr^2}\right) + \frac{R(r)}{r^2}\frac{d^2\Phi}{d\phi^2} = 0 \tag{2.4.3}$$

$$\iff \frac{r^2}{R}\left(\frac{1}{r}\frac{dR}{dr} + \frac{d^2R}{dr^2}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \tag{2.4.4}$$

$$\tag{2.4.5}$$

Since the first term is ϕ-independent, and the second term is r-independent, we can set them both equal to constants $\pm k^2$:

$$\frac{1}{R}\left(r\frac{dR}{dr} + r^2\frac{d^2R}{dr^2}\right) = k^2 \tag{2.4.6}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -k^2 \tag{2.4.7}$$

which can be rewritten as

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - k^2R = 0 \tag{2.4.8}$$

$$\frac{d^2\Phi}{d\phi^2} + k^2\Phi = 0 \tag{2.4.9}$$

The solution for is:

$$\Phi(\phi) = a\cos k\phi + b\sin k\phi \tag{2.4.10}$$

For the radial equation, let's try the ansatz $R(r) = r^n$ for some n:

$$r^n n(n-1) + r^n n - k^2 r^n = 0 \iff n = \pm k,\ k \neq 0 \tag{2.4.11}$$

hence r^k and r^{-k} form a fundamental set of solutions provided $k \neq 0$, so we know that $R(r) = Cr^k + Dr^{-k}$. If instead $k = 0$, then one of the basis solutions is constant, but what about the second solution? Using the method of variation of parameters, we can set $R(r) = cf(r)$ and thus:

$$r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} = 0 \tag{2.4.12}$$

$$\iff \frac{df}{dr} = \frac{E}{r} \tag{2.4.13}$$

$$\iff f(r) = E \ln r + F \tag{2.4.14}$$

Consequently:

$$\boxed{V(r, \phi) = A_0 \ln r + B_0 + \sum_{k=1}^{\infty} \left[r^k (A_k \cos k\phi + B_k \sin k\phi) + r^{-k}(C_k \cos k\phi + D_k \sin k\phi)\right]} \tag{2.4.15}$$

Potential outside long metal pipe in uniform E

Consider an infinitely long metal pipe inside an otherwise electric field $\mathbf{E} = E_0\hat{\mathbf{r}}$. The potential

Potential due to long metal pipe with charge density $\sigma(\phi)$

Consider an infinitely long metal cylinder of radius R with a charge $\sigma(\phi) = a \sin 5\phi$ glued on its surface.

The boundary conditions for this system are:

(i) $\sigma(\theta) = a \sin 5\phi$

(ii) $V \to 0$ as $r \to \infty$

For the potential outside, we can immediately rule out the terms $A_0 \ln r$ and r^k, since they blow up for large r. Similarly, for the potential inside, we can also rule out the term $A_0 \ln r$ which blows up at the origin, as well as the term with r^{-k} in the sum. Thus:

$$V_{in} = \sum_{k=1}^{\infty} r^k (A_k \cos k\phi + B_k \sin k\phi) \tag{2.4.16}$$

$$V_{out} = \sum_{k=1}^{\infty} r^{-k} (C_k \cos k\phi + D_k \sin k\phi) \tag{2.4.17}$$

Now:

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\bigg|_{r=R} = -\frac{a \sin 5\phi}{\varepsilon_0 r} \tag{2.4.18}$$

and since:

$$\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} = \sum_{k=1}^{\infty} -kR^{-k-1}(C_k \cos k\phi + D_k \sin k\phi) \tag{2.4.19}$$

$$\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = \sum_{k=1}^{\infty} kR^{k-1}(A_k \cos k\phi + B_k \sin k\phi) \tag{2.4.20}$$

we find that:

$$\sum_{k=1}^{\infty} k\left[R^{-k-1}(C_k \cos k\phi + D_k \sin k\phi) + R^{k-1}(A_k \cos k\phi + B_k \sin k\phi)\right] = \frac{a \sin 5\phi}{\varepsilon_0} \tag{2.4.21}$$

Simplifying a bit:

$$\sum_{k=1}^{\infty} k\left[R^{-k-1}(C_k \cos k\phi + D_k \sin k\phi) + R^{k-1}(A_k \cos k\phi + B_k \sin k\phi)\right] = \frac{a \sin 5\phi}{\varepsilon_0} \tag{2.4.22}$$

Due to the orthogonality of $\sin k\phi$ and $\cos k\phi$, we find that $C_k = D_k = 0$ and $A_k = B_k = 0$ for $k \neq 5$. If instead $k = 5$:

$$5\left[\frac{1}{R^6}(C_5 \cos 5\phi + D_5 \sin 5\phi) + R^4(A_5 \cos 5\phi + B_5 \sin 5\phi)\right] = \frac{a \sin 5\phi}{\varepsilon_0} \tag{2.4.23}$$

Therefore:

$$\begin{cases} \frac{D_5}{R^6} + R^4 B_5 = \frac{a}{5\varepsilon_0} \\ \frac{C_5}{R^6} + R^4 A_5 = 0 \end{cases} \implies \begin{cases} D_5 = \frac{aR^6}{5\varepsilon_0} - R^{10} B_5 \\ C_5 = -A_5 R^{10} \end{cases} \tag{2.4.24}$$

So, the potentials take the form of:

$$V_{in} = r^5(A_5 \cos 5\phi + B_5 \sin 5\phi) \tag{2.4.25}$$

$$V_{out} = \frac{1}{r^5}\left[- A_5 R^{10} \cos 5\phi + \left(\frac{aR^6}{5\varepsilon_0} - R^{10} B_5\right) \sin 5\phi\right] \tag{2.4.26}$$

Using the continuity of the potential at $r = R$:

$$R^5(A_5 \cos 5\phi + B_5 \sin 5\phi) = -A_5 R^5 \cos 5\phi + \left(\frac{aR}{5\varepsilon_0} - R^5 B_5\right) \sin 5\phi \tag{2.4.27}$$

$$A_5 \cos 5\phi + B_5 \sin 5\phi = -A_5 \cos 5\phi + \left(\frac{a}{5R^4\varepsilon_0} - B_5\right) \sin 5\phi \tag{2.4.28}$$

so that $A_5 = 0$. Instead:

$$2B_5 \sin 5\phi = \frac{a \sin 5\pi}{5R^4\varepsilon_0} \implies B_5 = \frac{a}{10R^4\varepsilon_0} \tag{2.4.29}$$

Finally:

$$V_{in} = \frac{ar^5}{10\varepsilon_0 R^4}\sin 5\phi \tag{2.4.30}$$

$$V_{out} = \frac{aR^6}{10\varepsilon_0 r^5}\sin 5\phi \tag{2.4.31}$$

2.5 Solution methods in spherical coordinates

Let us try to solve Laplace's equation in spherical coordinates, assuming azimuthal independence so that V is independent of ϕ. Then, Laplace's equation becomes:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \tag{2.5.1}$$

Again, using separation of variables, we seek solutions of the form:

$$V(r,\theta) = R(r)\Theta(\theta) \tag{2.5.2}$$

which yields:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0 \tag{2.5.3}$$

Now, since the first term is radial, and the second term is angular, we can set each to a constant, which for reasons that will become clear soon we set as $l(l+1)$:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) \tag{2.5.4}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1) \tag{2.5.5}$$

For the first, we can again use the ansatz $R = r^n$ for some n, then we quickly find that:

$$r^n(n-1)e^{n-2} + 2rnr^{n-1} = l(l+1)r^n \iff n = l, -(l+1) \tag{2.5.6}$$

so that the fundamental set of solutions is $R_1(r) = r^l, R_2(r) = \frac{1}{r^{l+1}}$. Hence the general solution may be written as the linear combination:

$$R(r) = Ar^l + \frac{B}{r^{l+1}} \tag{2.5.7}$$

The second equation is known as the **Rodriguez equation** (see mathematical methods volume). To simplify our lives, let's set $x = \cos\theta$, so that

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = \sin\theta\frac{d}{dx} \tag{2.5.8}$$

and thus (2.5.5) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right)+l(l+1)\Phi=0 \tag{2.5.9}$$

Let's consider the polynomial $p_l=(x^2-1)^l$, which satisfies:

$$(x^2-1)\frac{d}{dx}(x^2-1)^l=2xl(x_2-1)^l \iff p_l\frac{dp_l}{dx}=2xlp_l \tag{2.5.10}$$

Differentiating $l+1$ times:

$$\frac{d^{l+1}}{dx^{l+1}}\left((x^2-1)\frac{d}{dx}(x^2-1)^l\right)=\frac{d^{l+1}}{dx^{l+1}}(2xl(x^2-1)^l) \tag{2.5.11}$$

or using shorthand notation $D^l=\frac{d^l}{dx^l}$:

$$D^{l+1}(p_1Dp_l)=D^{l+1}(2xlp_l) \tag{2.5.12}$$

Now we can use Leibniz's formula:

$$D^n(fg)=\sum_{k=0}^{n}\binom{n}{k}(D^{n-k}f)(D^kg) \tag{2.5.13}$$

and note that $D^k(x^2-1)=0$ for $k\geq 3$ so that:

$$p_1D^{l+2}p_l+(l+1)Dp_1D^{l+1}p_l+(l+1)lD^2p_1D^lp_l=2l(xD^{l+1}p_l+(l+1)D^lp_l) \tag{2.5.14}$$

$$\iff (x^2-1)D^{l+2}p_l+2x(l+1)D^{l+1}p_l+l(l+1)D^lp_l=2l(xD^{l+1}p_l+(l+1)D^lp_l) \tag{2.5.15}$$

$$\iff 0=-2xD^{l+1}p_l+(1-x^2)D^{l+2}p_l+l(l+1)D^lp_l \tag{2.5.16}$$

$$\iff 0=D((1-x^2)D^{l+1}p_l)+l(l+1)D^lp_l=0 \tag{2.5.17}$$

$$\iff 0=\frac{d}{dx}\left((1-x^2)\frac{d^{l+1}}{dx^{l+1}}p_l\right)+l(l+1)\frac{d^l}{dx^l}p_l=0 \tag{2.5.18}$$

which is exactly the same as (2.5.9). Hence the solutions to the Rodriguez equation are the so-called **Legendre polynomials** which upon normalization become:

$$\boxed{P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l} \tag{2.5.19}$$

and thus substituting back $x=\cos\theta$:

$$\Theta(\theta)=P_l(\cos\theta) \tag{2.5.20}$$

We may therefore conclude that:

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)} \tag{2.5.21}$$

An important property of Legendre polynomials is the orthogonality condition:

$$\int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1}\delta_{lm} \tag{2.5.22}$$

Hence, if we express some function $f(\cos x)$ continuous over $[0,\pi]$ in terms of the Legendre polynomials:

$$f(\cos\theta) = \sum_l c_l P_l(\cos\theta) \tag{2.5.23}$$

then using Fourier's trick we can write that:

$$\int_0^{\pi} f(\cos\theta) P_m(\cos\theta) = \sum_l \left[c_l \int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\ \theta d\theta \right] \tag{2.5.24}$$

implying that:

$$\boxed{c_m = \frac{2m+1}{2} \int_0^{\pi} f(\cos\theta) P_m(\cos\theta) \sin\theta d\theta} \tag{2.5.25}$$

Another important property of the Legendre polynomials is that they are even for l even and odd for l odd, since:

$$\boxed{P_l(-x) = (-1)^l P_l(x)} \tag{2.5.26}$$

Spherical shell split in two charged hemispheres

Consider a conducting spherical shell of radius R split into a northern hemisphere held at a potential V and a southern hemisphere with uniform surface charge $-V$. The two hemispheres are divided by an infinitesimally thin insulating ring.

The potential of this configuration is in its most general form:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \tag{2.5.27}$$

Inside the shell, we must drop the $\frac{1}{r^l}$ term which diverges as $r \to 0$. Instead, outside the shell we must drop the term r^l which diverges as $r \to \infty$, so that:

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta),\ r \le R \tag{2.5.28}$$

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_{l+1}}{r^l} P_l(\cos\theta),\ r \ge R \tag{2.5.29}$$

Now since the potential is continuous at $r = R$, we have that:

$$\sum_{l=0}^{\infty} AR^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B}{R^{l+1}} P_l(\cos\theta) \tag{2.5.30}$$

and using the orthogonality of Legendre polynomials:

$$\sum_{l=0}^{\infty} \int_0^{\pi} A_l R^l P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta = \sum_{l=0}^{\infty} \int_0^{\pi} \frac{B_l}{R^{l+1}} P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta \tag{2.5.31}$$

$$A_l R^l = \frac{B_l}{R^{l+1}} \implies B_l = A_l R^{2l+1} \tag{2.5.32}$$

Moreover, we can specify the potential on the surface of the sphere:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V(\theta) = \begin{cases} V \text{ for } 0 < \theta < \frac{\pi}{2} \\ -V \text{ for } \frac{\pi}{2} < \theta < \pi \end{cases} \tag{2.5.33}$$

So we find that:

$$\sum_{l=0}^{\infty} \int_0^{\pi} A_l R^l P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta = \int_0^{\pi/2} V P_m(\cos\theta)\sin\theta d\theta - \int_{\pi/2}^{\pi} V P_m(\cos\theta)\sin\theta d\theta \tag{2.5.34}$$

so that:

$$\frac{2}{2m+1} A_m R^m = \int_0^{\pi/2} V P_m(\cos\theta)\sin\theta d\theta - \int_{\pi/2}^{\pi} V P_m(\cos\theta)\sin\theta d\theta \tag{2.5.35}$$

Substituting $x = \cos\theta$ and using the property that $P_l(-x) = (-1)^l P_l(x)$, we can write that:

$$\frac{2}{2m+1} A_m R^m = \int_0^1 V P_m(x) dx + \int_0^{-1} V P_m(x) dx \tag{2.5.36}$$

$$= \int_0^1 V P_m(x) dx + (-1)^m \int_0^{-1} V P_m(-x) dx \tag{2.5.37}$$

$$= \int_0^1 V P_m(x) dx - (-1)^m \int_0^{1} V P_m(x) dx \tag{2.5.38}$$

$$= [1 - (-1)^m] \int_0^1 V P_m(x) dx \tag{2.5.39}$$

so that all even terms vanish. The first two non-null contributions to the potential will therefore be $m = 1$ and $m = 3$.

Firstly

$$\frac{2}{3}A_1 R = 2\int_0^1 VP_1(x)dx = V \tag{2.5.40}$$

$$\implies A_1 = \frac{3V}{2R}, B_1 = \frac{3VR^2}{2} \tag{2.5.41}$$

Similarly:

$$\frac{2}{7}A_3 R^3 = 2\int_0^1 VP_3(x)dx = -\frac{V}{4} \tag{2.5.42}$$

$$\implies A_3 = -\frac{7V}{8R^3}, B_3 = -\frac{7VR^4}{8} \tag{2.5.43}$$

We can approximate the potential to the first two terms as:

$$V_{in}(r,\theta) \approx \frac{3Vr}{2R}\cos\theta - \frac{7Vr^3}{16R^3}(5\cos^3\theta - 3\cos\theta) \tag{2.5.44}$$

$$= \frac{Vr}{2R}\cos\theta\left(3 - \frac{7r^2}{8R^2}(5\cos^2\theta - 3)\right) \tag{2.5.45}$$

and:

$$V_{out}(r,\theta) \approx \frac{3VR^2}{2r^2}\cos\theta - \frac{7VR^4}{16r^4}(5\cos^3\theta - 3\cos\theta) \tag{2.5.46}$$

$$= \frac{VR^2}{2r^2}\cos\theta\left(3 - \frac{7R^2}{8r^2}(5\cos^2\theta - 3)\right) \tag{2.5.47}$$

We can also evaluate the induced surface charge:

$$\sigma_{ind} = -\varepsilon_0\left(\frac{\partial V_{out}}{\partial r}\bigg|_{r=R} - \frac{\partial V_{in}}{\partial r}\bigg|_{r=R}\right) \tag{2.5.48}$$

$$= \varepsilon_0\sum_{l=0}^{\infty}\left(\frac{(l+1)B_l}{R^{l+2}} + lA_l R^{l-1}\right)P_l(\cos\theta) \tag{2.5.49}$$

$$\approx \varepsilon_0\left[\left(\frac{2B_1}{R^3} + A_1\right)P_1(\cos\theta) + \left(\frac{4B_3}{R^5} + 3A_3R^2\right)P_3(\cos\theta)\right] \tag{2.5.50}$$

Hence:

$$\sigma_{ind} = \varepsilon_0\left[\frac{9V}{2R}P_1(\cos\theta) - \left(\frac{7V}{2R} + \frac{21V}{8R}\right)P_3(\cos\theta)\right] \tag{2.5.51}$$

and after some simplification:

$$\sigma_{ind} \approx \frac{\varepsilon_0 V}{R}\left(\frac{9}{2}P_1(\cos\theta) - \frac{49}{8}P_3(\cos\theta)\right) \tag{2.5.52}$$

Electrostatics in matter

3

3.1 Polarization

Most materials can be categorized as:

(i) **Conductors**: have unlimited supply of charges free to roam around.

(ii) **Dielectrics**: all charges are associated/fixed to an atom, and are not free to roam around.

Suppose we place a neutral atom in an electric field $\mathbf{E}$. The positively charged nucleus and the negative charged electron cloud will then separate under the influence of this field. If $\mathbf{E}$ is strong enough, then the electrons will be ripped from the atom's nucleus, and ionization occurs. Otherwise, there will be a balance between the attraction of the electron and nucleus, and the force due to $\mathbf{E}$, producing a dipole which is usually approximately proportional to the applied field:

$$\mathbf{p} = \alpha \mathbf{E} \tag{3.1.1}$$

Here we define α as the **atomic polarizability** of the atom in question.

Indeed, let us model an atom as a point nucleus of charge $+q$ and uniform spherical cloud $-q$ of radius a. Polarization will occur when $\mathbf{E} = \mathbf{E}'$, where $\mathbf{E}'$ is the attraction between the electron cloud and nucleus. If they are separated by d then:

$$E_e = \frac{qd}{4\pi\varepsilon_0 a^3} = E \tag{3.1.2}$$

and since the induced dipole moment is $p = qd$:

$$p = 4\pi\varepsilon_0 a^3 E \implies \boxed{\alpha = 4\pi\varepsilon_0 a^3} \tag{3.1.3}$$

is an approximate formula.

For more complicated systems, such as molecules, we need to use the polarizability tensor:

$$\mathbf{p} = \overleftrightarrow{\alpha}\mathbf{E} = \begin{pmatrix} \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{pmatrix} \tag{3.1.4}$$

where α_{ij} gives the component of polarizability that produces a dipole moment along i due to an electric field along j. For example, α_{xy} gives the component of the dipole moment in the x direction due to the electric field along y.

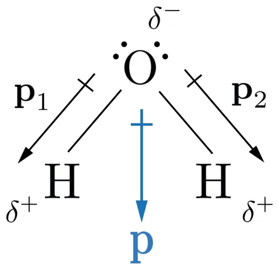


Figure 3.1. Polarity of water

Polar molecules, such as water, have a built-in permanent dipole. This is because oxygen is more electronegative than hydrogen, so the electron cloud tends to move closer to the oxygen atom than to the two hydrogen atoms. Overall, this creates two dipole moments for each oxygen-hydrogen interaction, which summed together give one overall dipole moment pointing down.

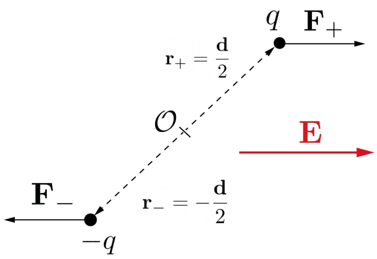


Figure 3.2. Torque on a dipole subject to external field

Usually, the dipole moments of polar molecules are randomly oriented. However, when we apply an external electric field, a net torque will be applied to each molecule:

$$\boldsymbol{\tau} = (\mathbf{r}_+ \times (q\mathbf{E})) + (\mathbf{r}_- \times (q\mathbf{E})) = q\mathbf{d} \times \mathbf{E} \tag{3.1.5}$$

assuming $\mathbf{F}_+ = \mathbf{F}_-$ so that:

$$\boxed{\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}} \tag{3.1.6}$$

We see that this torque tends to align each molecule's dipole moment along the applied electric field, producing a large scale dipole. If instead we have $\mathbf{F}_+ \neq \mathbf{F}_-$ then:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q\delta\mathbf{E} \tag{3.1.7}$$

and using Taylor's theorem:

$$\delta\mathbf{E} = d_x \frac{\partial \mathbf{E}}{\partial x} + d_y \frac{\partial \mathbf{E}}{\partial y} + d_z \frac{\partial \mathbf{E}}{\partial z} = (\mathbf{d} \cdot \nabla)\mathbf{E} \tag{3.1.8}$$

Hence $\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}}$. The torque on the dipole is then:

$$\boldsymbol{\tau} = \mathbf{r} \times (\mathbf{p} \cdot \nabla)\mathbf{E} \tag{3.1.9}$$

Now note that:

$$[\mathbf{r} \times (\mathbf{p} \cdot \nabla)\mathbf{E}]_i = \epsilon_{ijk} r_j p_l \partial_l E_k = \epsilon_{ijk} r_j p_l \partial_k E_l \tag{3.1.10}$$
$$= -\epsilon_{ikj} r_j p_l \partial_k E_l = [-\nabla \times ((\mathbf{p} \cdot \mathbf{E})\mathbf{r})]_i \tag{3.1.11}$$

where we used $\nabla \times \mathbf{E} = 0$. Hence, using the vector identity:

$$\nabla \times (\phi \mathbf{A}) = \phi(\nabla \times \mathbf{A}) + (\nabla \phi) \times \mathbf{A} \tag{3.1.12}$$

we find that:

$$\boldsymbol{\tau} = -\nabla \times ((\mathbf{p} \cdot \mathbf{E})\mathbf{r}) = -(\mathbf{p} \cdot \mathbf{E})\nabla \times \mathbf{r} - \nabla(\mathbf{p} \cdot \mathbf{E}) \times \mathbf{r} \tag{3.1.13}$$
$$= -\nabla(\mathbf{p} \times \mathbf{E}) \times \mathbf{r} \tag{3.1.14}$$
$$\implies \boldsymbol{\tau} = \mathbf{r} \times \nabla(\mathbf{p} \times \mathbf{E}) \tag{3.1.15}$$
$$\implies \mathbf{F} = \nabla(\mathbf{p} \times \mathbf{E}), \quad \text{and } U = -\mathbf{p} \cdot \mathbf{E} \tag{3.1.16}$$

3.2 Field of a polarized dielectric

So suppose we have polarized some dielectric by the methods explained in the previous section. How do we determine the electric field that it produces?

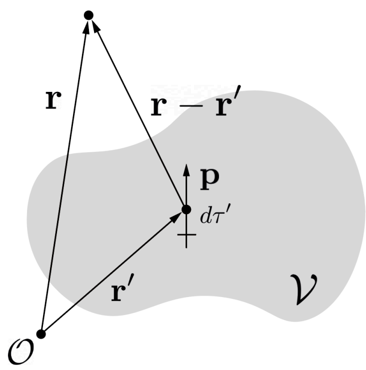


Figure 3.3. Geometry of dielectric

If we define polarization $\mathbf{P}$ as the dipole moment per unit volume so that $\mathbf{p} = \mathbf{P} d\tau'$ then the potential due to one dipole is:

$$dV(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{3.2.1}$$

then:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d\tau \tag{3.2.2}$$

and using $\nabla'\Big(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big) = \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$ then:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int_{\mathcal{V}} \mathbf{P}(\mathbf{r}') \cdot \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\right)\right) \tag{3.2.3}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\oint_{\mathcal{S}} \frac{\mathbf{P}\cdot d\mathbf{a}}{|\mathbf{r}-\mathbf{r}'|}}_{\text{due to } \sigma_b} - \underbrace{\int_{\mathcal{V}} \frac{1}{|\mathbf{r}-\mathbf{r}'}(\nabla'\cdot\mathbf{P})d\tau'}_{\text{due to } \rho_b}\right] \tag{3.2.4}$$

The first integral is due to the bound surface charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ and the second integral is due to the bound volume charge $\rho_b = -\nabla\cdot\mathbf{P}$. We can then write everything as:

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{\mathcal{S}} \frac{\sigma_b}{|\mathbf{r}-\mathbf{r}'|} da' + \int_{\mathcal{V}} \frac{\rho_b}{|\mathbf{r}-\mathbf{r}'|} d\tau'\right]} \tag{3.2.5}$$

However, in the real world we cannot model the individual dipoles as perfect dipoles like we did in the previous derivation. Instead, we must model them as physical dipoles. Consequently 3.2.5 only applies in the real world for the field outside from the dielectric, where the dipole approximation applies. Inside, we must be more careful.

Evaluating the exact microscopic field inside a dielectric would be very very difficult, almost impossible most of the time. Instead we can evaluate the **macroscopic field**, defined as the average field over regions that are small enough to contain significant fluctuations, but large enough to ignore microscopic bumps.

The microscopic field is more formally:

$$\langle\mathbf{E}\rangle = \langle\mathbf{E}_{in}\rangle + \langle\mathbf{E}_{out}\rangle \tag{3.2.6}$$

where $\langle\mathbf{E}_{in}\rangle$ and $\langle\mathbf{E}_{out}\rangle$ are the average field due to charges inside and outside respectively.

The average field over a sphere (centered at $\mathbf{r}$) due to charges outside was proven to be the field at $\mathbf{r}$ in the previous chapter. We model the dipoles outside as perfect dipoles and find:

$$\langle V_{out}\rangle = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}\setminus\mathcal{S}} \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d\tau' \tag{3.2.7}$$

We have also proven that the average field produced by the charges inside is:

$$\langle\mathbf{E}_{in}\rangle = -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3} \tag{3.2.8}$$

where $\mathbf{p} = \frac{4\pi R^3 \mathbf{P}}{3}$ is the total dipole moment. Consequently:

$$\langle \mathbf{E}_{in} \rangle = -\frac{\mathbf{P}}{3\varepsilon_0} \tag{3.2.9}$$

Since the sphere is small enough to ignore any uninteresting fluctuations, the term left out of the integral in 3.2.7 is:

$$\frac{1}{4\pi\varepsilon_0} \int_{\mathcal{S}} \frac{\mathbf{P} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau' \tag{3.2.10}$$

which corresponds to the field due to a uniformly polarized sphere, which is exactly $\langle \mathbf{E}_{in}(\mathbf{r}\rangle = -\frac{\mathbf{P}}{3\varepsilon_0}$, the term that $\langle V_{in} \rangle$ introduces. So:

$$\langle V \rangle = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau' \tag{3.2.11}$$

so we were calculating the average macroscopic field all along. This relies on the fact that no matter how bumpy the microscopic fluctuations get, we can smooth them out by a spherical distribution of spherical dipoles [1].

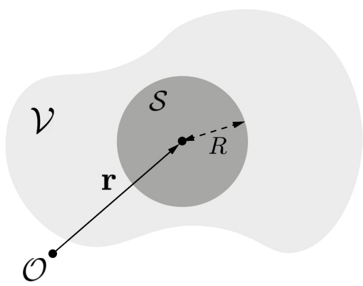
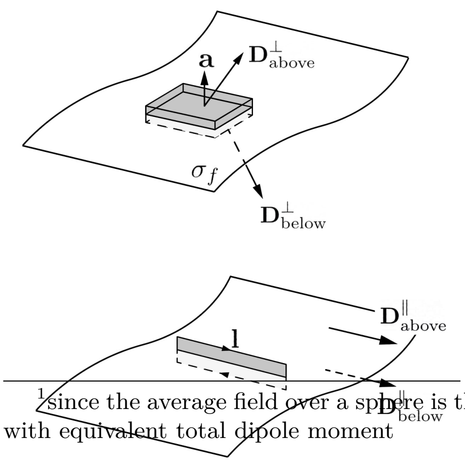


Figure 3.4. Macroscopic field calculation

3.3 Electric displacecement vector and Gauss' law



For a dielectric:

$$\rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f \tag{3.3.1}$$

so that:

$$\varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho_f \tag{3.3.2}$$

If we define $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ as the **electric displacement vector** then:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f} \tag{3.3.3}$$

[1] since the average field over a sphere is the same as the field at the center of a uniformly polarized sphere with equivalent total dipole moment

which is the revised form of Gauss' law for dielectrics.

We can reformulate the boundary conditions for a boundary between two dielectrics. Let us firstly note that

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P} \tag{3.3.4}$$

so that taking the line integral around the path straddling the boundary surface:

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}} \tag{3.3.5}$$

Similarly, taking the surface integral for the thin Gaussian pillbox:

$$\mathbf{D}^{\perp}_{\text{above}} - \mathbf{D}^{\perp}_{\text{below}} = \sigma_f \hat{\mathbf{n}}_{12} \tag{3.3.6}$$

3.4 Linear dielectrics

We must now pose a distinction between different types of dielectrics:

(i) **linear**: if χ_e is independent of E, so that E and P are directly proportional.

(ii) **isotropic**: if χ_e is independent of the direction of $\mathbf{E}$.

(iii) **homogeneous**: if $\nabla \chi_e = 0$.

Generally, most materials are linear for low fields. If the fields get too large, the dielectric may break down, that is, it becomes conductive.

Liquids and gases are generally isotropic, whereas most solids are anisotropic. This is due to the strong intermolecular forces that cause solids to be rigid, and limit the rotation of their constituent molecules to align with the applied field.

Materials that are linear, isotropic and homogeneous satisfy the relation:

$$\boxed{\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}} \tag{3.4.1}$$

Then:

$$\mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E} \equiv \varepsilon \mathbf{E} \tag{3.4.2}$$

where $\varepsilon = \varepsilon_0(1 + \chi_e)$ is the permittivity of the dielectric, and $\varepsilon_r = (1 + \chi_e) = \frac{\varepsilon}{\varepsilon_0}$ is the relative permittivity of the dielectric. We call ξ_e the electric susceptibility.

In free space, there is no polarization, so that $\chi_e = 0$ and hence $\varepsilon = \varepsilon_0$ as expected.

Also, in a linear dielectric:

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\frac{\varepsilon_0 \chi_e}{\varepsilon} \mathbf{D} \right) = -\frac{\chi_e}{1 + \chi_e} \rho_f \tag{3.4.3}$$

So if there is no free charge, then there is no bound volume charge, Any charge must therefore reside on the surface of the dielectric.

We can therefore use all our tools for solving BVPs we explained in the previous chapter. Specifically, we can use the boundary conditions:

$$\varepsilon_{above}\frac{\partial V_{above}}{\partial n} - \varepsilon_{below}\frac{\partial V_{above}}{\partial n} = -\sigma_f \tag{3.4.4}$$

$$V_{above} = V_{below} \tag{3.4.5}$$

Let us consider a dielectric with polarization $\mathbf{P}$. The energy of this configuration is no longer the work required to construct the dielectric charge by charge, since this does not take into account the work done to produce $\mathbf{P}$.

We instead define the energy of a dielectric as the work needed to produce its polarization by constructing it free charge by free charge.

So the infinitesimal work done to increase the charges by $\delta\rho_f$ is:

$$\delta W = \int_{\mathcal{V}} V(\mathbf{r})\delta\rho_f d\tau \tag{3.4.6}$$

but $\delta\rho_f = \nabla\cdot\delta\mathbf{D}$ so that:

$$\delta W = \int_{\mathcal{V}} V\nabla\cdot(\delta\mathbf{D})d\tau' \tag{3.4.7}$$

Note that by the divergence theorem:

$$\int_{\mathcal{V}} V\nabla\cdot(\delta\mathbf{D})d\tau' = \int_{\mathcal{V}} \nabla\cdot(V\mathbf{D})d\tau' + \int_{\mathcal{V}} \mathbf{E}\cdot\delta\mathbf{D}d\tau' \tag{3.4.8}$$

$$= \int_{\mathcal{S}} V\mathbf{D}\cdot d\mathbf{S} + \int_{\mathcal{V}} \mathbf{E}\cdot\delta\mathbf{D}d\tau' \tag{3.4.9}$$

For a localized dielectric, we can take the integral over all of space, so that the first surface integral vanishes giving:

$$\delta W = \int_{\mathcal{V}} \mathbf{E}\cdot\delta\mathbf{D}d\tau' \tag{3.4.10}$$

For a linear dielectric, we have that:

$$\delta W = \int \delta\mathbf{D}\cdot\mathbf{E}d\tau = \int \varepsilon\delta\mathbf{E}\cdot\mathbf{E}d\tau = \frac{\varepsilon}{2}\int \delta E^2 d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}d\tau \tag{3.4.11}$$

hence:

$$\boxed{W = \frac{1}{2}\int_{\mathbb{R}^3} \mathbf{D}\cdot\mathbf{E}d\tau} \tag{3.4.12}$$

3.5 Force on dielectrics

Suppose we partially fill a parallel plate capacitor with some dielectric. In the real world, we cannot model the field of the capacitor as being completely normal to the dielectric. Instead, there are some fringing fields at the boundaries of the capacitor. It must be the case that fringing fields are present, since otherwise the line integral around a path that is partially outside the capacitor would not be null, violating $\nabla \times \mathbf{E} = 0$. These fringing fields are responsible for a horizontal force pushing the dielectric into the capacitor.

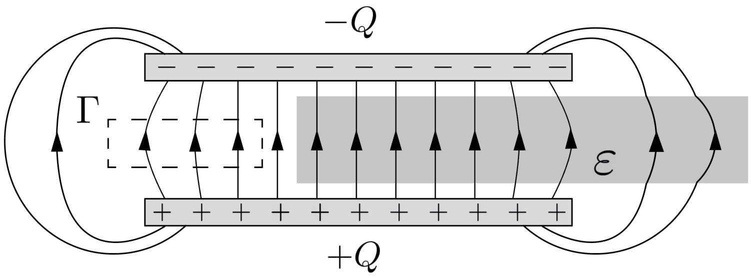


Figure 3.5. Fringing fields of a parallel plate capacitor

Unfortunately, these fields are very hard to calculate, but we can work around this problem. Indeed, suppose the capacitor configuration had some energy W, and suppose I pull the dielectric out by some dx. Then, the force I have to exert F_{me} must be equal in magnitude to the force F due to the fringing fields. Hence:

$$\delta W = F_{me}\delta x \implies F = -\frac{dW}{dx} \tag{3.5.1}$$

and since the energy stored in a capacitor is $C = \frac{1}{2}CV^2$ we find that:

$$F = -\frac{1}{2}\frac{d}{dx}(CV^2) = -\frac{1}{2}\frac{d}{dx}\left(\frac{Q^2}{C}\right) \tag{3.5.2}$$

If we set the charge on the capacitor to be constant (instead of the potential to be constant) then:

$$F = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx} \tag{3.5.3}$$

We could have also set the potential to be constant instead of the charge, by connecting the capacitor to a battery. However, we would then have to take into account the work done by the battery to maintain the constant potential while moving the dielectric. Then:

$$dW = -F dx + V dQ \implies F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} \tag{3.5.4}$$

so again:

$$\boxed{F = \frac{1}{2}V^2\frac{dC}{dx}} \tag{3.5.5}$$

Part II

Magnetostatics

Fundamentals of Magnetostatics

4

4.1 Electric Current

An electric current is defined as charge in motion, carried by so-called **charge carriers**. The **current intensity** through a wire is the amount of charge passing through in unit time, and is measured in Amperes (A). We say that the current is **steady when it is time-independent**, the current density is therefore the same at every point in the wire at all times.

Definition: ELECTRIC CURRENT

Electric current is formed by charge carriers in motion.
Steady electric currents have a homogeneous current density, and have no time dependence.

Consider a wire through which n particles per cubic meter on average, are all moving with the same velocity $\mathbf{v}$ carrying a charge Q. Then, the amount of charge passing through the wire of cross section A in Δt will be the amount of charge present in the prism shown below, which is its volume times the charge density:

$$nQ\mathbf{A} \cdot \underbrace{(\mathbf{v}\Delta t)}_{width} \tag{4.1.1}$$

Hence the current through the cross section will be:

$$I = nQ\mathbf{A} \cdot \mathbf{v} \tag{4.1.2}$$

If instead we have a variety of charge carriers with different charge, population density and velocities then the resulting current is:

$$\boxed{I = \mathbf{A} \cdot \sum_k n_k Q_k \mathbf{v}_k} \tag{4.1.3}$$

and we shall define the current density to be:

$$\mathbf{J} = \sum_k n_k Q_k \mathbf{v}_k \tag{4.1.4}$$

In the case of a more complex surface $\mathcal{S}$, the sum turns into a surface integral, and we find that:

$$\boxed{I = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{A}} \tag{4.1.5}$$

We define a steady current to be one where the current density $\mathbf{J}$ remains constant everywhere in time, so that $\nabla \cdot \mathbf{J} = 0$. By the divergence theorem:

$$I = \oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{A} = \oint_{\mathcal{V}} \nabla \cdot \mathbf{J} = 0 \tag{4.1.6}$$

so the flux of the current density through any closed surface is zero. **what comes in comes out**. This also means that the integral over any open surface is invariant as long as the boundary remains the same. Indeed, consider two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{C}$. Then, taking $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$:

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{A} = \oint_{\mathcal{S}_1} \mathbf{J} \cdot d\mathbf{A} - \oint_{\mathcal{S}_2} \mathbf{J} \cdot d\mathbf{A} = 0 \implies \oint_{\mathcal{S}_1} \mathbf{J} \cdot d\mathbf{A} = \oint_{\mathcal{S}_2} \mathbf{J} \cdot d\mathbf{A} \tag{4.1.7}$$

where the minus sign is because $\mathcal{S}_2$ has opposite orientation to $\mathcal{S}_1$.

Current density

The current through a wire with n particles per cubic meter on average, each with charge Q_k, drift velocity v_k and abundance n_k is:

$$I = \mathbf{A} \cdot \sum_k n_k Q_k \mathbf{v}_k \tag{4.1.8}$$

where $\mathbf{A} = A\hat{\mathbf{n}}$ is the cross section vector. For more complex surfaces:

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{A} \tag{4.1.9}$$

4.2 Lorentz Force Law

It is a fact that two current-carrying wires will feel an attractive or repulsive force depending on the direction of the current. For parallel currents, a repulsive force will be present, whereas for opposite currents, an attractive force will be exerted.

This phenomenon can be explained by the Lorentz force law: $\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$

This is a fundamental law applying to all charged particles in any circumstance. Because of this, we can use it to define the magnetic field:

Definition: MAGNETIC FIELD

If we can find a vector field $\mathbf{B}$ such that, when substituted into the Lorentz force law, gives the measured magnetic force on any moving point charge, then we define this field as the **magnetic field**.

If a line charge λ travels down a wire at speed $\mathbf{v}$, then there is a current:

$$\mathbf{I} = \lambda \mathbf{v} \tag{4.2.1}$$

The magnetic force on a segment of current carrying wire is:

$$\mathbf{F}_{mag} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda \, dl = \int (\mathbf{I} \times \mathbf{B}) \, dl \tag{4.2.2}$$

Since $\mathbf{I}$ and $d\mathbf{l}$ point in the same direction, we can then write:

$$\mathbf{F}_{mag} = \int I(d\mathbf{l} \times \mathbf{B}) \tag{4.2.3}$$

Similarly, for charge flowing over a surface, we have a surface current density:

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} = \sigma \mathbf{v} \tag{4.2.4}$$

and the magnetic force experienced will be:

$$\mathbf{F}_{mag} = \int_{\mathcal{S}} \mathbf{K} \times \mathbf{B} \, da \tag{4.2.5}$$

Finally, for volume charges we have the volume current density:

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} = \rho \mathbf{v} \tag{4.2.6}$$

and the magnetic force experienced will be:

$$\boxed{\mathbf{F}_{mag} = \int_{\mathcal{V}} \mathbf{J} \times \mathbf{B} \quad d\tau} \tag{4.2.7}$$

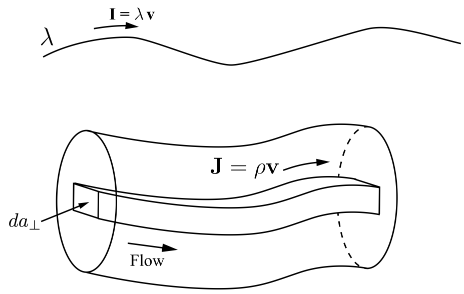


Figure 4.1. Line and Volume Charge Densities

Lorenz Force Law

The total electromagnetic force on a particle of charge q moving with velocity $\mathbf{v}$ is:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4.2.8}$$

More specifically, for a charge distribution the magnetic force can be expressed as:

$$\mathbf{F}_{mag} = \int_{\mathcal{V}} \mathbf{J} \times \mathbf{B} \;\; d\tau \tag{4.2.9}$$

4.3 Local charge conservation

Note that the total current flowing through a distribution can be defined on one hand as:

$$I = -\frac{d}{dt}\int_{\mathcal{V}} \rho \; d\tau \tag{4.3.1}$$

and on the other hand as:

$$\mathbf{I} = \int_{\mathcal{S}} \mathbf{J} \; da = \int_{\mathcal{V}} \nabla \cdot \mathbf{J} \; d\tau. \tag{4.3.2}$$

Equating the two expressions yields the **continuity equation**:

$$\boxed{-\frac{d}{dt}\int_{\mathcal{V}} \rho \; d\tau = \int_{\mathcal{V}} \nabla \cdot \mathbf{J} \; d\tau \implies \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}} \tag{4.3.3}$$

This equation states the **local conservation of charge**, that is, if charge is conserved locally in any region unless there is some current through the boundary of the region compensating for aggregation/rarefaction of charges.

Moreover, since in magnetostatics, we will be working in the regime of steady currents (no charges piling up somewhere):

$$\frac{\partial \rho}{\partial t} = 0, \;\; \frac{\partial \mathbf{J}}{\partial t} = 0 \tag{4.3.4}$$

the continuity equation then implies:

$$\nabla \cdot \mathbf{J} = 0 \tag{4.3.5}$$

which is analogous to the incompressible flow equation in fluid dynamics. It says that the surface integral of $\mathbf{J}$ over any closed surface must be zero. If current is flowing, it must flow through the surface, it can't be flowing in or out of the surface, for if this were the case charge would start accumulating somewhere.

Another interesting implication is that the derivative of the dipole moment of a configuration is:

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\int \rho \mathbf{r}\, d\tau = \int \frac{\partial \rho}{\partial t}\mathbf{r}\, d\tau \tag{4.3.6}$$

and:

$$\int_{\mathcal{V}} \nabla \cdot (x_i \mathbf{J})\, d\tau = \int_{\mathcal{V}} x_i \nabla \cdot \mathbf{J}\, d\tau + \int_{\mathcal{V}} \mathbf{J} \cdot (\nabla x_i)\, d\tau \tag{4.3.7}$$

Moreover, by the divergence theorem:

$$\int_{\mathcal{V}} \nabla \cdot (x_i \mathbf{J})\, d\tau = \int_{\mathcal{S}} x_i \mathbf{J} \cdot d\boldsymbol{S} = 0 \tag{4.3.8}$$

since all charges and currents are contained within $\mathcal{V}$, $\mathbf{J} \cdot d\boldsymbol{S} = 0$. Hence:

$$\int_{\mathcal{V}} x_i \nabla \cdot \mathbf{J}\, d\tau = -\int_{\mathcal{V}} \mathbf{J} \cdot (\nabla x_i)\, d\tau$$

$$\int_{\mathcal{V}} x_i \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \mathbf{J} \cdot \hat{\mathbf{x}}_i\, d\tau \implies \int_{\mathcal{V}} x_i \frac{\partial \rho}{\partial t}\, d\tau = \int_{\mathcal{V}} J_{x_i}$$

and repeating this for the other x_i coordinates yields:

$$\int_{\mathcal{V}} \mathbf{J}\, d\tau = \frac{d\mathbf{p}}{dt} \tag{4.3.9}$$

Continuity equation and time-derivative of dipole moment

For a charge density ρ and current density $\mathbf{J}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{4.3.10}$$

known as the **continuity equation**.

4.4 Biot-Savart Law

To evaluate the magnetic field produced by a steady current, we use the Biot-Savart law:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{C}} \frac{\mathbf{I} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, dl' = \frac{\mu_0 I}{4\pi}\int_{\mathcal{C}} \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}} \tag{4.4.1}$$

where $\mathcal{C}$ is the path of the current, μ_0 the permeability of free space, $d\mathbf{I}$ an infinitesimal line element tangent to the path, and $\mathbf{r}-\mathbf{r}'$ is the vector in the direction from the source to $\mathbf{r}$ (see below).

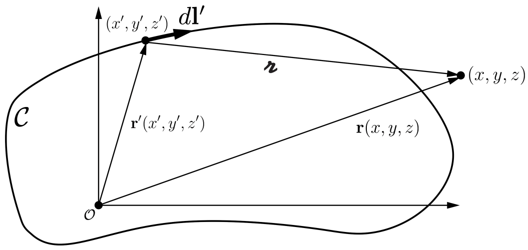


Figure 4.2. Geometry of Biot-Savart's Law

Similarly, for surface and volume currents:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{S}} \frac{\mathbf{K}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, da \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d\tau' \tag{4.4.2}$$

The **magnetostatic force** on a current density $\mathbf{J}^*$ due to $\mathbf{J}$ is then:

$$\boxed{\mathbf{F} = \int_{\mathcal{V}} \mathbf{J}^* \times \mathbf{B}\, d\tau = \frac{\mu_0}{4\pi}\int_{\mathcal{V}^*}\int_{\mathcal{V}} \frac{\mathbf{J}^*(\mathbf{r}) \times (\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}'))}{|\mathbf{r}-\mathbf{r}'|^3}\, d\tau' d\tau} \tag{4.4.3}$$

For example, the magnetic field created by an infinite wire is:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi s} \tag{4.4.4}$$

and for an infinite solenoid carrying current I with n turns per unit length, the magnetic field along its axis is:

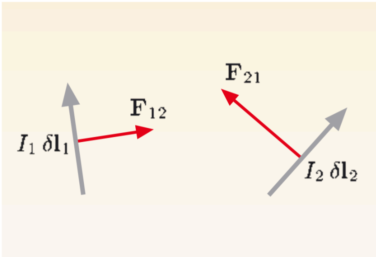
$$\mathbf{B}(\mathbf{r}) = n\mu_0 I \tag{4.4.5}$$

Strategy. (Evaluating magnetic field produced by a current distribution)
To evaluate the magnetic field produced using the **Biot-Savart law**:

1. Evaluate $|d\mathbf{I} \times (\mathbf{r}-\mathbf{r}')|$, and transform any unknown variables such as length into polar coordinates.
2. Evaluate $\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}$ in terms of the aforementioned coordinates.
3. Use these quantities to evaluate $d\mathbf{B}$, and then integrate along the entire distribution.
4. Use the right hand rule and the symmetry of the distribution to determine the direction of the magnetic field.

4.5 Paradoxes with the Biot-Savart law

It may appear as though the Biot-Savart law violates Newton's third law. Indeed, consider two current elements as shown below:



The forces they exert on each other are not opposite in direction, so what's wrong? In reality, Biot-savart can't be applied to isolated current elements, since for steady state currents we need closed circuits. So, technically we would need to evaluate the force over all the current elements. If we did so, we would find that they do indeed point in opposite directions.

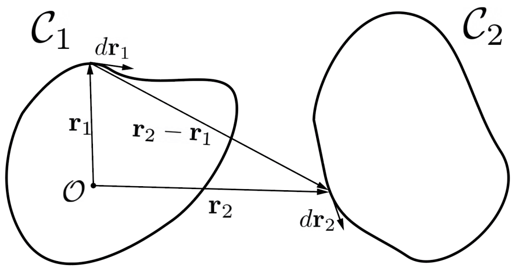


Figure 4.3. Geometry of two interacting current loops

Indeed, we know that:

$$\mathbf{F}_{12} = \oint_{\mathcal{C}_2} I_2(d\mathbf{r}_2 \times \mathbf{B}_1) \tag{4.5.1}$$

$$= \oint_{\mathcal{C}_2}\oint_{\mathcal{C}_1} \frac{\mu_0 I_1 I_2}{4\pi} d\mathbf{r}_1 \times \left(\frac{d\mathbf{r}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}\right) \tag{4.5.2}$$

For sake of simplicity, let $\boldsymbol{\mathscr{r}} = \mathbf{r}_2 - \mathbf{r}_1$, then we can use the BAC-CAB rule to write that:

$$d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \boldsymbol{\mathscr{r}}) = (d\mathbf{r}_1 \cdot \boldsymbol{\mathscr{r}})d\mathbf{r}_2 - (d\mathbf{r}_1 \cdot d\mathbf{r}_2)\boldsymbol{\mathscr{r}} \tag{4.5.3}$$

so that:

$$\mathbf{F}_{21} = -\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} \frac{\mu_0 I_1 I_2}{4\pi} d\mathbf{r}_1 \times \left(\frac{d\mathbf{r}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}\right) \tag{4.5.4}$$

$$= -\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} \frac{\mu_0 I_1 I_2}{4\pi} \frac{(d\mathbf{r}_1 \cdot \boldsymbol{\mathscr{r}})d\mathbf{r}_2 - (d\mathbf{r}_1 \cdot d\mathbf{r}_2)\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} \tag{4.5.5}$$

However using Stoke's theorem (taking the curl with respect to $\mathbf{r}_2$) we find that the first term disappears:

$$\oint_{\mathcal{C}_\infty}\oint_{\mathcal{C}_\in} \frac{(d\mathbf{r}_1 \cdot \hat{\boldsymbol{\mathscr{r}}})d\mathbf{r}_2}{\mathscr{r}^2} = \oint_{\mathcal{C}_\infty}\oint_{\mathcal{S}_\in} d\mathbf{r}_1 \cdot \cancelto{0}{\left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)} d\mathbf{r}_2 = \mathbf{0} \tag{4.5.6}$$

leaving:

$$\mathbf{F}_{21} = \oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} \frac{\mu_0 I_1 I_2}{4\pi} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) \tag{4.5.7}$$

Using a similar logic we can also prove that:

$$\mathbf{F}_{12} = -\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} \frac{\mu_0 I_1 I_2}{4\pi} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) = -\mathbf{F}_{21} \tag{4.5.8}$$

as desired.

Another seemingly apparent contradiction is the fact that the Biot-savart law makes no mention of instantaneity. According to this law, turning on a current on Earth would have an immediate effect on some current on Mars. Again, we can solve this using the fact that steady currents have always been "on". If there were a moment when the current is switched on then it would not be steady/time-independent. Therefore Biot-Savart's law is not applicable anymore.

4.6 No-monopole Law and Ampere's Law

Consider the Biot Savart law:

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, d\tau' \\
\implies \nabla \cdot \mathbf{B} &= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\right) d\tau' \\
&= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \cdot \cancelto{0}{(\nabla \times \mathbf{J})} - \mathbf{J} \cdot \cancelto{0}{\left(\nabla \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\right)} d\tau' = 0
\end{aligned}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{4.6.1}$$

where we took the divergence over unprimed variables. This is the equivalent of Gauss' Law in magnetostatics, and has a significant physical interpretation. Indeed, using the divergence theorem:

$$\boxed{\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0} \tag{4.6.2}$$

over any closed surface $\mathcal{S}$. This means that **there are no magnetic monopoles**, unlike electrostatics where we have positive and negative charges. Indeed, if there were magnetic monopoles, then they would act as source terms and create a non-zero magnetic flux.

No-monopole law

There are no magnetic monopoles (as of the writing of this document), so that:

$$\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0 \iff \nabla \cdot \mathbf{B} = 0 \tag{4.6.3}$$

over any closed surface $\mathcal{S}$.

Taking the curl instead:

$$\begin{aligned}
\nabla \times \mathbf{B} &= \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \ d\tau' \\
&= \frac{\mu_0}{4\pi} \int \left[\mathbf{J} \left(\nabla \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) - (\mathbf{J} \cdot \nabla) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right] \ d\tau'
\end{aligned}$$

Note however, that since $\nabla = -\nabla'$ (i.e. $\frac{\partial f(x-x')}{\partial x} = -\frac{\partial f(x-x')}{\partial x'}$), then:

$$\int_{\mathcal{V}} (\mathbf{J} \cdot \nabla) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ d\tau' = -\int (\mathbf{J} \cdot \nabla') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ d\tau' \tag{4.6.4}$$

The x-component is (assuming a steady current [1] so that $\nabla \cdot \mathbf{J} = 0$):

$$\begin{aligned}
&\int_{\mathcal{V}} (\mathbf{J} \cdot \nabla') \left(\frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \ d\tau' \\
&= \int_{\mathcal{V}} \nabla' \cdot \left(\frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J} \right) - \left(\frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \cancelto{0}{(\nabla' \cdot \mathbf{J})} d\tau' \\
&= \int_{\mathcal{S}} \frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J} \ da' \\
&= 0
\end{aligned}$$

for a sufficiently large closed surface $\mathcal{S}$ completely enclosing the current. We can repeat this for all components to conclude that:

$$\int_{\mathcal{V}} (\mathbf{J} \cdot \nabla) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ d\tau' = 0$$

[1] the case for non-steady currents will be treated quasistatic fields

and that therefore:

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J} \left(\nabla \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \ d\tau' = \frac{\mu_0}{4\pi} \int \mathbf{J} (4\pi \delta^3 (\mathbf{r} - \mathbf{r}')) \ d\tau' \tag{4.6.5}$$

since $\nabla \cdot (\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}) = 4\pi\delta^3(\mathbf{r} - \mathbf{r}')$. Therefore, we reach **Ampere's Law**:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}} \tag{4.6.6}$$

which can be rewritten in integral form similarly to Gauss' Law as:

$$\boxed{\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}} \tag{4.6.7}$$

over any open surface $\mathcal{S}$ with boundary $\mathcal{C}$. Here, I_{enc} is the current that skewers/pierces our surface $\mathcal{C}$.

For example, the magnetic field due to a toroidal coil (a circular coil around which a long wire is tightly wrapped) has a circumferential magnetic field. Taking an Amperian loop around the axis of the coil, the magnetic field inside has magnitude $\frac{\mu_0 NI}{2\pi s}$ and outside there is no magnetic field (since the current coming out cancels the current coming in).

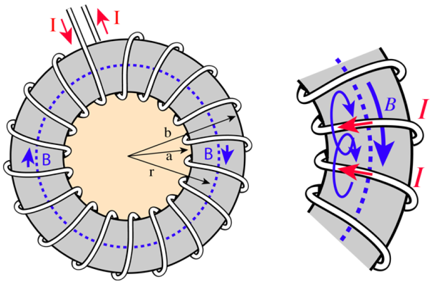


Figure 4.4. Current-Carrying Toroidal Coil and its Magnetic Field

Ampere's Law

For steady currents, **Ampere's law** holds:

$$\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \iff \nabla \times \mathbf{E} = \mu_0 \mathbf{J} \tag{4.6.8}$$

4.7 Symmetry arguments part 2

As in the case of Gauss' law, Ampere's law can only be used in symmetric configurations. However, it may seem like the arguments used previously lack rigour, as they rely on failure to find alternatives.

A more rigorous approach is to find a symmetry operation (see Group theory in Mathematical methods volume), that leaves the source of the EM field unaltered.

We then see how this operation acts on a given coordinate system. By the symmetry principle, if the action of the operation is unrecognizable on the source, then any coordinate that is altered must be null.

We must now differentiate symmetry arguments for electric fields and magnetic fields.

This is because $\mathbf{B}$ at any given point is not a real real vector, but a pseudovector. This means that it almost acts as a vector, but not quite. Indeed, consider the following reflection:

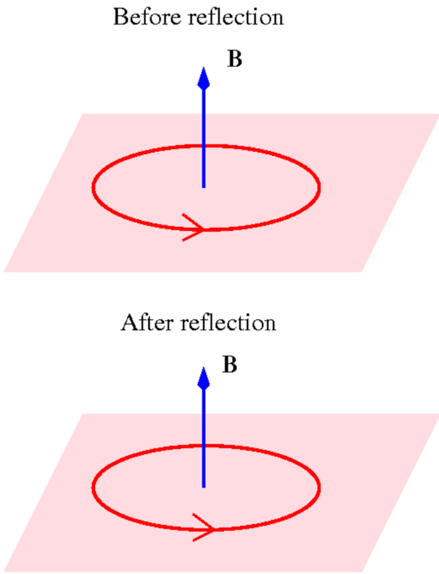


Figure 4.5. Electron orbit around nucleus

We see that if we had reflected the magnetic field in the plane, it would have been pointing down, which goes against what the right hand rule would say for the given current ring.

We must therefore reverse $\mathbf{B}$, the component perpendicular to the field after reflecting it. Any parallel component however, undergoes reflection without any problem.

This is because the Biot-Savart law defines the magnetic field as a cross product. If we let $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ then:

$$c_x = (a_y b_z - a_z b_y) \tag{4.7.1}$$
$$c_y = (a_z b_x - a_x b_z) \tag{4.7.2}$$
$$c_z = (a_x b_y - a_y b_x) \tag{4.7.3}$$

If we now perform a reflection in the xy-plane, $a_z \to -a_z$ and $b_z \to -b_z$. So, we find:

$$c'_x = -(a_y b_z - a_z b_y) = -c_x \tag{4.7.4}$$
$$c'_y = -(a_z b_x - a_x b_z) = -c_y \tag{4.7.5}$$
$$c_z = (a_x b_y - a_y b_x) = c_z \tag{4.7.6}$$

as required, only the component perpendicular to the plane is unaltered.

In the case of the electric field instead, reflection of course only reverses any component perpendicular to the plane of reflection. This is as expected, since $\mathbf{E}$ is a true/polar vector.

In general, we can state the following:

Reflection of electromagnetic fields

To reflect an electric field in a plane, we reverse the component of the field perpendicular to this plane at every point in the plane.
To reflect an magnetic field in a plane, we reverse the component of the field parallel to this plane at every point in the plane.

For example, let us use symmetry arguments to find the form of the magnetic field at some point P due to a cylindrical current density.

We can deduce from the translational symmetry along the z-axis and rotational symmetry about the z-axis that there is no dependence on ϕ nor z. We can therefore write:

$$\mathbf{B}(\mathbf{r}) = B_r(r)\hat{\mathbf{e}}_r + B_\phi(r)\hat{\mathbf{e}}_\phi + B_z(r)\hat{\mathbf{e}}_z \tag{4.7.7}$$

We now consider a reflection in the plane containing P and the z-axis. The current distribution remains the same (since it is uniform), however the r and z components, which are parallel to the plane, are reversed. Due to the symmetry principle, we can deduce that $B_r = B_z = 0$. Hence:

$$\mathbf{B}(\mathbf{r}) = B_\phi(r)\hat{\mathbf{e}}_\phi \tag{4.7.8}$$

We cannot repeat this argument with a reflection in the xy-plane, since this would reverse the current distribution as well.

If we repeat this argument with a reflection in the xy-plane, the current distribution and $B_\phi(r)\hat{\mathbf{e}}_\phi$ are both reversed, as required.

4.8 Magnetic Vector Potential

The **magnetic vector potential A** in magnetostatics is defined as:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}, \;\; \nabla \cdot \mathbf{A} = 0} \tag{4.8.1}$$

One might immediately wonder: *why add $\nabla \cdot \boldsymbol{A} = 0$?*.

Recall that the electric potential had an innate vagueness, since we could add any constant to V without altering its gradient, $\mathbf{E}$.

Similarly, we may add an irrotational (curl-less) vector field to $\mathbf{A}$ without altering its curl $\mathbf{B}$. Such vector fields can always be expressed as the gradient of some scalar function, allowing us to write the following transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla\xi \tag{4.8.2}$$

called a **gauge transformation**.

Vector analysis allows us to exploit this ambiguity. Indeed, it always allows us to find a solenoidal (divergence-less) vector potential $\nabla \cdot \mathbf{A} = 0$.

To see why this is true, assume that we have found some vector field $\mathbf{A}$ reproducing the correct magnetic field $\nabla \times \mathbf{A} = \mathbf{B}$, but unfortunately gives $\nabla \cdot \mathbf{A} = \zeta$. We can use a specific gauge transformation $\mathbf{A}' = \mathbf{A}_2 + \nabla\xi$ such that:

$$\nabla \cdot \mathbf{A}' = \zeta + \nabla^2\xi = 0 \implies \nabla^2\xi = -\zeta \tag{4.8.3}$$

Note that the above equation is the Poisson equation, and as we have shown previously we can always find a function ξ satisfying this equation. This solution is:

$$\xi = \frac{1}{4\pi}\int \frac{\zeta}{|\mathbf{r}-\mathbf{r}'|^2}\, d\tau \tag{4.8.4}$$

Hence it is always possible to find an appropriate gauge transformation giving:

$$\nabla \cdot \mathbf{A}' = 0 \tag{4.8.5}$$

Taking the curl of $\mathbf{A}$:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \cancelto{0}{\nabla(\nabla \cdot \mathbf{A}} - \nabla^2\mathbf{A}) = \mu_0\mathbf{J} \tag{4.8.6}$$

so that:

$$\nabla^2\mathbf{A} = \mu_0\mathbf{J} \tag{4.8.7}$$

This is identical to three Poisson's equations:

$$\nabla_i^2\mathbf{A}_i = \mu_0 J_i \tag{4.8.8}$$

and has solution (assuming $\mathbf{J}$ does not extend to infinity):

$$\boxed{A_i = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{J_i}{|\mathbf{r}-\mathbf{r}'|^2} d\tau'} \tag{4.8.9}$$

which combined give:

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|^2} d\tau'} \tag{4.8.10}$$

If instead the current extends to infinity, then we can notice that:

$$\oint_{\partial\mathcal{S}} \mathbf{A} \cdot d\mathbf{l} = \int_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = \Phi \tag{4.8.11}$$

Typically, the magnetic potential points in the same direction as the current.

Definition: MAGNETIC VECTOR POTENTIAL

The magnetic vector potential satisfies the pair of PDEs:

$$\mathbf{B} = \nabla \times \mathbf{A}, \; \nabla \cdot \mathbf{A} = 0 \tag{4.8.12}$$

In the case of a steady current not extending to infinity:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' \tag{4.8.13}$$

4.9 Boundary Conditions

Just like the electric field, magnetic fields also suffer from a discontinuity for surface currents. However, unlike electrostatics, this time the discontinuity lies in the tangential component.

Consider an arbitrary surface current, and let us apply the solenoidal nature of the magnetic field:

$$\oint \mathbf{B} \cdot d\mathbf{a} = \oint \mathbf{B}_{above} \cdot d\mathbf{a} - \oint \mathbf{B}_{below} \cdot d\mathbf{a} = 0 \implies B^{\perp}_{above} = B^{\perp}_{below} \tag{4.9.1}$$

since the sides of the pillbox give no contribution ($d\mathbf{a}$ points upwards). Let us now take an Amperian loop running perpendicular to the current:

$$\oint \mathbf{B} \cdot d\mathbf{l} = (B^{\|}_{above} - B^{\|}_{below})l = \mu_0 I_{enc} = \mu_0 K l \tag{4.9.2}$$

so:

$$B^{\|}_{above} - B^{\|}_{below} = \mu_0 K. \tag{4.9.3}$$

These two conditions can be summarised into:

$$\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \tag{4.9.4}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface oriented upwards. This condition can be rewritten as:

$$\boxed{\frac{\partial \mathbf{A}_{above}}{\partial n} - \frac{\partial \mathbf{A}_{below}}{\partial n} = -\mu_0 \mathbf{K}} \tag{4.9.5}$$

Boundary conditions for surface currents

The boundary condition over a surface current are:

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \iff \hat{n} \times (\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}}) = \mu_0 \mathbf{K} \tag{4.9.6}$$

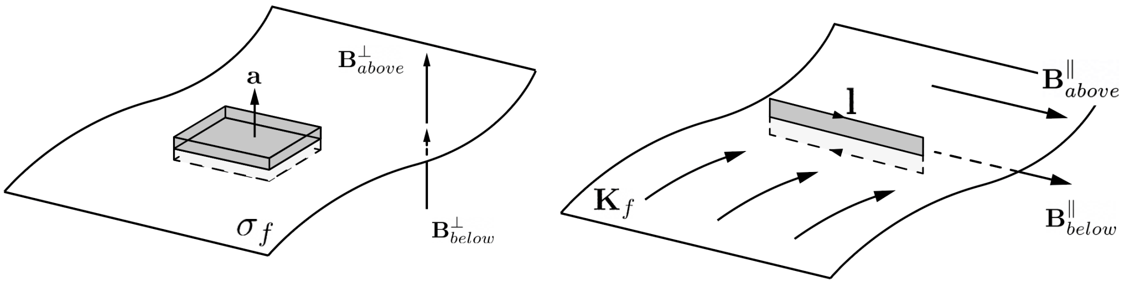


Figure 4.6. Boundary conditions for Magnetostatic fields

4.10 Magnetic Dipoles

We previously found that:

$$\frac{1}{\mathbf{r}-\mathbf{r}'} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha) \tag{4.10.1}$$

from which it follows that:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{l}}{\mathbf{r}-\mathbf{r}'} = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n P_n(\cos\alpha)d\mathbf{l}' \tag{4.10.2}$$

where the first term, the monopole term, is zero since:

$$\oint d\mathbf{l}' = 0. \tag{4.10.3}$$

The dipole term instead simplifies to:

$$\mathbf{A}_{dip} = \frac{\mu_0 I}{4\pi r^3}\oint r'\cos\alpha d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^3}\oint \hat{\mathbf{r}}\cdot\mathbf{r}' d\mathbf{l}' \tag{4.10.4}$$

Now we can use the identity:

$$\oint(\mathbf{c}\cdot\mathbf{r})d\mathbf{l} = \int d\mathbf{a}'\times\mathbf{c} \tag{4.10.5}$$

where $\mathbf{c}$ is a constant vector. To prove it, consider:

$$\oint_{\mathcal{C}}(\mathbf{r}\cdot\mathbf{r}')\mathbf{g}\cdot d\mathbf{l} \tag{4.10.6}$$

where $\mathbf{g}$ is yet another constant vector. Then:

$$\oint_{\mathcal{C}} (\mathbf{r}\cdot\mathbf{r}')\mathbf{g}\cdot d\mathbf{l} = \int_{\mathcal{S}} \nabla\times(\mathbf{g}(\mathbf{r}\cdot\mathbf{r}'))\cdot d\mathbf{S} \tag{4.10.7}$$

$$= \int_{\mathcal{S}} \varepsilon_{ijk}\partial'_j(g_k r_l r'_l) dS_i \tag{4.10.8}$$

$$= \int_{\mathcal{S}} \varepsilon_{ijk} g_k r_l dS_i \tag{4.10.9}$$

$$= -\int_{\mathcal{S}} \mathbf{g}\cdot(\mathbf{r}\times d\mathbf{S}) \tag{4.10.10}$$

This is true for all constant vectors $\mathbf{g}$ we can write:

$$\oint (\mathbf{r}\cdot\mathbf{r}')d\mathbf{l} = \int d\mathbf{S}\times\mathbf{r} \tag{4.10.11}$$

Hence:

$$\mathbf{A}_{dip} = \frac{\mu_0 I}{4\pi r^3}\left(\oint d\mathbf{S}\times\mathbf{r}'\right) \tag{4.10.12}$$

or in our case:

$$\oint \hat{\mathbf{r}}\cdot\mathbf{r}' d\mathbf{l}' = -\mathbf{r}\times\oint d\mathbf{S}' \tag{4.10.13}$$

Finally, we find that:

$$\boxed{\mathbf{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}, \quad \mathbf{m} = I\int d\mathbf{a}'} \tag{4.10.14}$$

Taking the curl then gives:

$$\boxed{\mathbf{B}_{dip} = \frac{\mu_0}{4\pi}\frac{[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}} \tag{4.10.15}$$

Finally, consider a magnetic dipole with moment $\mathbf{m}$ placed in an external magnetic field $\mathbf{B}$. Then, a torque $\boldsymbol{\tau}$ will be exerted on the dipole about a point $\mathcal{O}$:

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times\int_{\mathcal{C}} I d\mathbf{l}\times\mathbf{B} = \int_{\mathcal{C}} I\left[(\mathbf{r}\cdot\mathbf{B})d\mathbf{l} - \mathbf{B}\,\cancelto{0}{(\mathbf{r}\cdot d\mathbf{l})}\right] = \int_{\mathcal{C}} I d\mathbf{a}\times\mathbf{B} = \mathbf{m}\times\mathbf{B} \tag{4.10.16}$$

Hence, the work done on the dipole is:

$$W = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{l} = \int_{C} F\sin\theta dl = \int_{\frac{\pi}{2}}^{\theta} F\sin\theta r d\theta = \int_{\frac{\pi}{2}}^{\theta} \tau d\theta = mB\cos\theta = \mathbf{m}\cdot\mathbf{B} \tag{4.10.17}$$

These results will be justified more rigorously in the following section accounting for more general current distributions.

Definition: MAGNETIC DIPOLES

The magnetic dipole moment $\mathbf{m}$ and the corresponding magnetic are defined as:

$$\mathbf{m} = I\int d\mathbf{a}',\ \mathbf{B}_{dip} = \frac{\mu_0}{4\pi}\frac{[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}{r^3} \tag{4.10.18}$$

4.11 Magnetic multipoles

For a more general current distribution, the ith component of the potential is:

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\left(\frac{J_i(\mathbf{r}')}{r} + \frac{J_i(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}')}{r^3}\right)d\tau' \tag{4.11.1}$$

As in the case of a line current, the monopole term disappears. Indeed, consider:

$$\partial_j(J_j r_i') = \cancelto{0}{(\partial_j J_j) r_i'} + J_i = J_i \tag{4.11.2}$$

where the first term vanishes due to the continuity equation $\nabla\cdot\mathbf{J} = \partial_j J_j = 0$. Thus:

$$\frac{1}{r}\int J_i d\tau' = \frac{1}{r}\int \partial_j(J_j r_i')d\tau' = \int (J_j r_i')dS' = 0 \tag{4.11.3}$$

assuming the current distribution is localized.

In a similar fashion:

$$\partial_j(J_j r_i' r_k') = (\partial_j J_j) r_k r_k' + J_k r_i' = J_i r_k' + J_k r_i' \tag{4.11.4}$$

implies that:

$$\int \partial_j(J_j r_i' r_k')d\tau' = \int (J_i r_k' + J_k r_i')dS' = 0 \implies J_i r_k' = -J_k r_i' \tag{4.11.5}$$

Hence:

$$\int J_i r_j r_j' d\tau' = \int \frac{r_j}{2}(J_i r_j' - J_j r_i')d\tau \tag{4.11.6}$$

$$= \frac{1}{2}\int (J_i(\mathbf{r}\cdot\mathbf{r}') - r_i'(\mathbf{J}\cdot\mathbf{r}))d\tau' \tag{4.11.7}$$

$$= \frac{1}{2}\int (\mathbf{r}\times(\mathbf{J}\times\mathbf{r}'))_i d\tau' \tag{4.11.8}$$

where we used the BAC-CAB rule $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{b}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$. Thus:

$$\int \mathbf{J}(\mathbf{r}\cdot\mathbf{r}')d\tau = -\frac{1}{2}\int \mathbf{r}\times(\mathbf{J}\times\mathbf{r}')d\tau \tag{4.11.9}$$

Consequently the dipole term becomes:

$$\mathbf{A}_{dip}(\mathbf{r}) = -\frac{\mu_0}{4\pi r^3}\frac{1}{2}\mathbf{r} \times \int \mathbf{J} \times \mathbf{r}' d\tau' \tag{4.11.10}$$

Magnetic multipoles

The dipole term of a current distribution is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad \mathbf{m} = \frac{1}{2}\int \mathbf{J} \times \mathbf{r}' d\tau' \tag{4.11.11}$$

where $\mathbf{m}$ is the **dipole moment**.

Finally, let us consider the interaction of dipoles with external magnetic fields.

The Lorentz force experienced by a dipole immersed in a field $\mathbf{B}$ is:

$$\mathbf{F} = \int \mathbf{J}_{dip} \times \mathbf{B} \tag{4.11.12}$$

Since the dipole is a current localized at $\mathbf{R}$ we may use the taylor expansion:

$$\mathbf{B}(\mathbf{R}) + (\mathbf{r} - \mathbf{R} \cdot \nabla)\mathbf{B}(\mathbf{R}) \tag{4.11.13}$$

where the differentiation occurs with respect to $\mathbf{R}$. Therefore:

$$\mathbf{F} = \int \mathbf{B}(\mathbf{R}) \times \cancelto{0}{\int \mathbf{J} d\tau'} + \int \mathbf{J} \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R}) d\tau' + (\mathbf{r}_0 \cdot \nabla)\mathbf{B}(\mathbf{R}) \times \cancelto{0}{\int \mathbf{J} d\tau'} \tag{4.11.14}$$

$$= \int \mathbf{J} \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R}) d\tau' \tag{4.11.15}$$

Now we perform some index sorcery, we wish to prove that:

$$\mathbf{J} \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R}) = -\nabla \times ((\mathbf{r} \cdot \mathbf{B}(\mathbf{R})\mathbf{J}) \tag{4.11.16}$$

The LHS can be written as:

$$\mathbf{J} \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R}) = \epsilon_{ijk} J_j r_l (\partial_l B_k) \tag{4.11.17}$$

whereas the RHS is:

$$-\nabla \times ((\mathbf{r} \cdot \mathbf{B}(\mathbf{R})\mathbf{J}) = \epsilon_{ijk} r_l (\partial_k B_l) J_j = \epsilon_{ijk} r_l J_j (\partial_l B_k) \tag{4.11.18}$$

Now we can write:

$$\mathbf{J} \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R}) + \nabla \times ((\mathbf{r} \cdot \mathbf{B}(\mathbf{R})\mathbf{J}) = \epsilon_{ijk} J_j r_l \underbrace{(\partial_l B_k - \partial_l B_k)}_{\nabla \times \mathbf{B}} \tag{4.11.19}$$

Since the dipole is localized, we can assume that the external magnetic field is constant over the dipole and that therefore:

$$\nabla \times \mathbf{B} = 0 \tag{4.11.20}$$

so that:

$$\mathbf{J} \times (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R}) + \nabla \times ((\mathbf{r} \cdot \mathbf{B}(\mathbf{R})\mathbf{J}) = 0 \tag{4.11.21}$$

as desired. Hence:

$$\mathbf{F} = -\nabla \times \int (\mathbf{r} \cdot \mathbf{B}(\mathbf{R}))\mathbf{J} d\tau' \tag{4.11.22}$$

We may now use 4.13.9 replacing $\mathbf{r} \leftrightarrow \mathbf{B}$ to find that:

$$\mathbf{F} = -\frac{1}{2}\nabla \times \left(\mathbf{B}(\mathbf{R}) \int \mathbf{J} \times \mathbf{r}' d\tau'\right) = \nabla \times (\mathbf{B} \times \mathbf{m}) \tag{4.11.23}$$

Finally, we make use of the vector identity:

$$\nabla \times (\mathbf{B} \times \mathbf{m}) = \mathbf{B}(\nabla \cdot \mathbf{m}) - \mathbf{m}(\nabla \cdot \mathbf{B}) + \nabla(\mathbf{B} \cdot \mathbf{m}) = \nabla(\mathbf{B} \cdot \mathbf{m}) \tag{4.11.24}$$

In conclusion we have found that:

$$\mathbf{F} = \nabla(\mathbf{B} \cdot \mathbf{m}) \tag{4.11.25}$$

and thus using $\mathbf{F} = -\nabla U$ and write down the potential energy of the dipole as:

$$U = -\mathbf{m} \cdot \mathbf{B} \tag{4.11.26}$$

Force, torque and potential energy of magnetic dipoles

The force on a magnetic dipole of moment $\mathbf{m}$ immersed in a field $\mathbf{B}$ is:

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{4.11.27}$$

so that its associated potential energy is:

$$U = -\mathbf{m} \cdot \mathbf{B} \tag{4.11.28}$$

The torque on it, instead, is:

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B} \tag{4.11.29}$$

4.12 Magnetic monopoles

We have seen that the no-monopole law, as the name suggests, states that there do not exist magnetic charges g:

$$\mathbf{B} = \frac{g}{4\pi r^2}\hat{\mathbf{r}} \tag{4.12.1}$$

Thus, if one were to split a bar magnet in two, each remnant will have its own north and south poles. It is impossible to produce a single north/south pole.

Furthermore, we have also seen in Volume 1: Quantum Mechanics that we are obliged to utilize the vector potential $\mathbf{A}$ when evaluating the Hamiltonian operator of a particle in an electromagnetic field. If magnetic monopoles were to exist that we would no longer be able to write $\mathbf{B} = \nabla \times \mathbf{A}$, but rather $\mathbf{B} = \nabla V_m$.

There seems to be a lot of evidence pitted against the existence of such magnetic monopoles, were it not for a beautiful theory formulated by the great P.A.M. Dirac.

In 1931, Dirac posited that if magnetic monopoles did in fact exist, then this would imply electric charge quantization:

$$ge = 2\pi\hbar n \tag{4.12.2}$$

known as the **quantization condition**.

4.13 Magnetic fields due to biological currents

So far we have been interested in quite artificial current distributions. However, often times in real life we are interested in magnetic fields due to complex current sources, such as the magnetic fields generated by biological currents.

Just reading this paragraph is a sign that electrical activity is occurring in your brain. It is therefore of great interest to be able to map the currents associateed with these electrical signals in organs such as the heart and brain.

The most common ways to do this are through the electrocardiogram (ecg) and electroencephalogram (eeg), which monitor the heart and brain's electrical activity. Nerves and muscle cells, much like batteries, act as small sources of energy which generate micro-currents through electric fields. One can measure the potential differences by placing electrodes on the skin. This method however is not very reliable, since it requires good electrical contact between the electrodes and the skin.

It is much more practical to measure the magnetic field generated by the currents.

Magnetostatics in Matter

5

5.1 Magnetization

Magnetization occurs when the individual magnetic dipoles inside a material get aligned by an external magnetic field, inducing a net magnetic moment in the material.

This chapter will present the formalism of magnetostatic fields in matter. Most of our results will be strikingly similar to their electrostatic counterparts as we shall soon see.

The main different between polarized media and magnetized media is that, although polarization is always parallel to the applied field, this is no longer the case with magnetization. Instead we can classify materials as:

(i) paramagnets: the magnetization is parallel to the applied magnetic field

(ii) diamagnets: the magnetization is opposite to the applied magnetic field

(iii) ferromagnets: the magnetization is retained after the applied field is turned off

The first two are relatively easy to explain, whereas the last is more intricate and will be reserved for later.

Let us see what processes can lead to paramagnetic and diamagnetic phenomena.

5.2 How does magnetization occur?

The process leading to paramagnetism has already been clarified in the previous chapter. We summarize our results below.

When a magnetic dipole is subject to an external magnetic field, a torque is exerted on it:

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B} \tag{5.2.1}$$

This torque tends to align the magnetic moment of the dipole along the direction of the magnetic field, indeed when $\mathbf{m} \parallel \mathbf{B}$ the torque is zero.

One might therefore expect paramagnetism to occur with most materials. However, it turns out that, due to the Pauli exclusion principle, electrons in atoms are paired so that they have opposing spin (magnetic moment). This neutralizes the paramagnetic effect.

It follows that paramagnetism most often occurs with atoms with an odd number of electrons, where the unpaired electrons exhibits a net moment along the applied field.

Let us now turn to the question of diamagnetic magnetization.

We can describe this phenomenon by analyzing an electron's orbit around the nucleus. Assume the orbit has radius R, so that the period of revolution is $T = \frac{2\pi R}{v}$. Because the electron is orbiting at very very high speeds, it will look like a steady current:

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R} \tag{5.2.2}$$

so that the total dipole moment of the electron is:

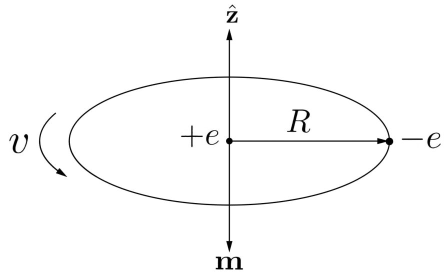


Figure 5.1. Electron orbit around nucleus

$$\mathbf{m} = -\frac{1}{2}evR\hat{\mathbf{z}} \tag{5.2.3}$$

This dipole moment is of course affected by an external magnetic field, whose torque causes the plane of orbit of the spin to tilt. This tilt however is very small, so the paramagnetic contribution here is almost negligible.

However, the electron also speeds up or slows down. Indeed, in absence of the external field the only centripetal force is the Coulomb force:

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = \frac{m_e v^2}{R} \tag{5.2.4}$$

Assume we now quickly turn on a magnetic field acting perpendicular to the plane of orbit. Then, assuming the radius of the orbit is virtually unchanged and remains circular:

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} + ev'B = \frac{m_e v'^2}{R} \tag{5.2.5}$$

Substituting (5.2.4) into the above:

$$ev'B = \frac{m_e}{R}(v'+v)(v'-v) \tag{5.2.6}$$

Assuming the change is very small:

$$ev'B = \frac{m_e}{R}(2v')(\Delta v) \implies \Delta v = \frac{eRB}{2m_e} \tag{5.2.7}$$

so the electron speeds up if $\mathbf{B}$ points up, and slows down if it points down.

This change in speed causes a change in the dipole moment:

$$\Delta\mathbf{m} = -\frac{1}{2}e\Delta v R\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B} \tag{5.2.8}$$

which is opposite to the direction of the applied field.

Hence, in the presence of a magnetic field each atom gains a small extra moment antiparallel to the field, leading diamagnetic phenomena. However, this effect is very small.

In contrast to paramagnetism, diamagnetism most often occurs in atoms with even number of electrons, where the former phenomenon is less pronounced.

Of course, this derivation is quite crude, and there are virtually no reasons to believe that the orbit of an electron is/remains circular. A more truthful derivation of diamagnetism can only stem from quantum mechanics.

5.3 Field due to magnetization

We have seen two possible phenomena that can lead to magnetization. We describe the magnetic dipole moment per unit volume induced in a material by the magnetization:

$$\mathbf{M}d\tau = \mathbf{m} \tag{5.3.1}$$

and is analogous to $\mathbf{P}$ in electrostatics. The vector potential of one perfect dipole $\mathbf{m}$ is

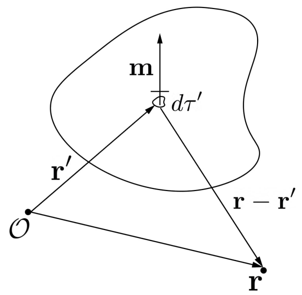


Figure 5.2. Coordinates and vectors of magnetized material

given by:

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r'}|^2} \tag{5.3.2}$$

For a material with magnetization $\mathbf{M}$, independently of how it got it, will have the same potential, but we must substitute $\mathbf{m}\to\mathbf{M}d\tau'$ and sum over each infinitesimal volume element

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{M}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r'}|^2}d\tau' \tag{5.3.3}$$

Just as in the case of the field of magnetized objects, we can use the identity:

$$\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} \tag{5.3.4}$$

to write that:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \left[\mathbf{M}(\mathbf{r}') \times \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] d\tau' \tag{5.3.5}$$

Now, since:

$$\nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left[\nabla' \times \mathbf{M}(\mathbf{r}') - \mathbf{M}(\mathbf{r}') \times \left(\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] \tag{5.3.6}$$

we may write that:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{\mathcal{V}} \frac{1}{\mathbf{A}(\mathbf{r})}(\nabla' \times \mathbf{M}(\mathbf{r}'))d\tau' - \int_{\mathcal{V}} \nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\tau'\right)\right] \tag{5.3.7}$$

We can use the the vector calculus identity:

$$\int_{\mathcal{V}} \nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\tau'\right) = -\int_{\mathcal{S}} \frac{\mathbf{M}(\mathbf{r}') \times d\mathbf{a}'}{|\mathbf{r}-\mathbf{r}'|} \tag{5.3.8}$$

to cast the second volume integral into a surface integral:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{\mathcal{V}} \frac{\overbrace{(\nabla' \times \mathbf{M}(\mathbf{r}')}^{\mathbf{J}_b}}{|\mathbf{r}-\mathbf{r}'|} d\tau' + \int_{\mathcal{S}} \overbrace{\left(\frac{\mathbf{M}(\mathbf{r}') \times d\mathbf{a}'}{|\mathbf{r}-\mathbf{r}'|}\right)}^{\mathbf{K}_b}\right] \tag{5.3.9}$$

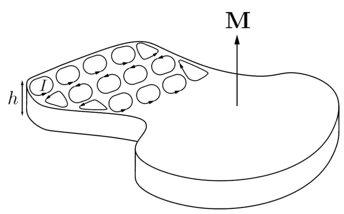
We see that the first integral looks like the potential of a volume current $\mathbf{J}_b$, whereas the second integral looks like the potential of a surface current $\mathbf{K}_b$. We can thus define the bound volume and surface currents due to magnetization $\mathbf{M}$ as:

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M}, \quad \text{and} \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}} \tag{5.3.10}$$

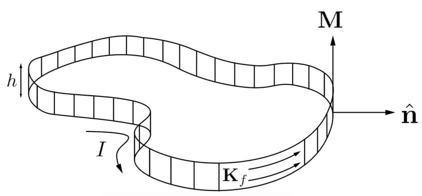
which produce the vector potential of a magnetized object.

5.4 How do bound currents arise?

There are interesting physical interpretations for the bound currents we have derived.

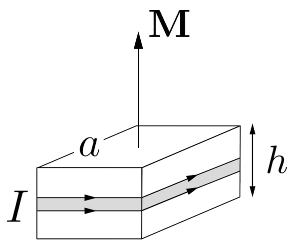


Consider a thin plate of uniformly magnetized material, containing several small current loops we model as dipoles.



Since the magnetization is uniform, one would expect the magnetic moment in the interior to cancel out, since every current going to the right must have a neighbor going to the left.

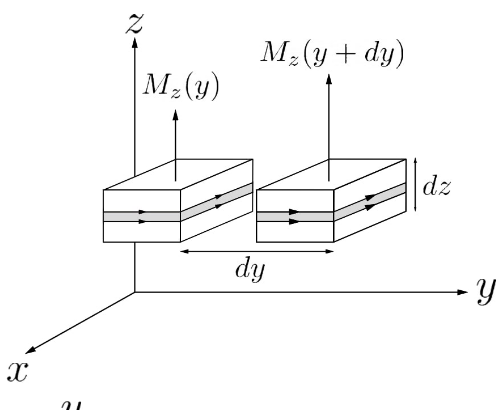
This does not happen at the boundary of the material, where there is no "adjacent loop". Hence, we can model the entire material as a ribbon of current:

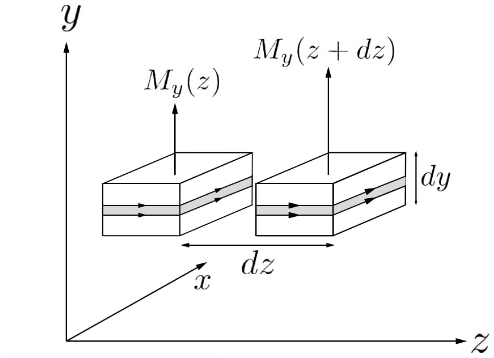


Suppose a current I runs around the ribbon, due to a magnetization $\mathbf{M}$. Let us model each dipole as loops of area a and thickness h. Then the moment of this loop is $m = M(ah)$, but $m = Ia$ so that $I = Mh$. The surface current is then the current flowing per unit length, that is, $K_b = \frac{I}{h} = M$. It follows that:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \tag{5.4.1}$$

as expected.





If instead the magnetization is non-uniform, then the cancellation of internal currents no longer happens. Consider two adjacent volume elements of the material.

Due to the non-uniformity of the magnetization, there must be some difference in the dipole of the two, say $M_z(y)$ and $M_z(y+dy)$. The surface in which the two chunks are joined therefore has two opposing currents in the x direction, the net result is an infinitesimal current:

$$dI_x^y = (M_z(y+dy) - M_z(y))dz = \frac{\partial M_z}{\partial y} dydz \tag{5.4.2}$$

Similarly, taking two chunks, and examining their magnetization in the y direction:

$$dI_x^z = -(M_y(z+dz) - M_y(z))dy = -\frac{\partial M_z}{\partial y} dydz \tag{5.4.3}$$

The same reasoning cannot be applied for non-uniform magnetization in the x direction, since the surface joining two chunks aligned along the x-axis has a current along the y-direction, and must therefore contribute to I_y not I_x.

The corresponding volume current density $(J_b)_x = \frac{dI_x}{da^\perp}$ therefore:

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \tag{5.4.4}$$

Repeating this argument for the current densities in the y and z directions we find that:

$$\mathbf{J}_b = \nabla \times \mathbf{M} \tag{5.4.5}$$

as expected.

One final test of consistency may be applied. Note that any steady current must have vanishing divergence. In the case of bound currents, the divergence of a curl always vanishes, so it is indeed true that:

$$\nabla \times \mathbf{J}_b = 0 \tag{5.4.6}$$

5.5 The auxiliary field and Ampere's law

The current that we have used in the previous chapter, $\mathbf{J}$, was really the sum of two currents, the bound current $\mathbf{J}_b$, and any other type of current, which shall be called free current, denoted by $\mathbf{J}_f$.

We may use this to cast Ampere's law into a different form containing only free currents. Indeed:

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \nabla \times \mathbf{M} \tag{5.5.1}$$

so that:

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f \tag{5.5.2}$$

The quantity in the brackets is called the **auxiliary field, H**:

$$\boxed{\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \quad \text{and} \quad \nabla \times \mathbf{H} = \mathbf{J}_f} \tag{5.5.3}$$

Unlike **D**, which is far less useful than **E**, **H** turns out to be more practical in laboratories than **B**. This is because when performing experiments, one most often measures the currents running through a material, free currents. The currents induced by the resulting magnetization are not measured, and hence one can more easily determine **H** than **B**.

Instead, in electrostatics **D** depends on the details of the materials used. When performing experiments one usually measures the potential difference in a capacitor for example. This determines **E**, but not **D**.

Also note that one cannot simply replace **B** with $\mu_0\mathbf{H}$ and total currents by free currents. It seems like:

$$\nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{H} \tag{5.5.4}$$

However, the curl alone does not determine a vector field, one must also know the divergence. Indeed, $\nabla \cdot \mathbf{B} = 0$ is an empirical law, always true, whereas:

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \tag{5.5.5}$$

which is not necessarily null. Only when the magnetization is uniform is it appropriate for such a replacement to be applied.

Consider for example a uniformly polarized bar magnet, with magnetization parallel to its axis. There is no free current here, one may be led to conclusion that $\mathbf{H} = 0$ so that $\mathbf{B} = \mu_0\mathbf{M}$ inside the magnet. Although it is true that the curl of the auxiliary field disappears everywhere, the divergence does not. Consequently, it is nonsense to say that $\mathbf{H} = 0$. In this specific case $\nabla \cdot \mathbf{M} \neq 0$ at the boundary between the magnet and its surroundings.

Using (5.5.5) we may also rewrite the boundary conditions:

$$H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below}) \tag{5.5.6}$$

Instead (5.5.3) we have that:

$$\mathbf{H}^{\parallel}_{above} - \mathbf{H}^{\parallel}_{below} = \mathbf{K}_f \times \hat{\mathbf{n}} \tag{5.5.7}$$

5.6 Linear media

It is very common for materias to acquire a magnetization proportional to the applied field, when the latter is not too strong. Traditionally however we express the proportionality between **M** and **H** instead of **B**:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}} \tag{5.6.1}$$

where χ_m is called the magnetic susceptibility. Materials satisfying (5.6.1) are called **linear media**, and also satisfy:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} \equiv \mu\mathbf{H} \tag{5.6.2}$$

where $\mu \equiv \mu_0(1 + \chi_m)$ is the permeability of the material. Of course, in vacuum χ_m so that μ_0 is the permeability of free space.

5.7 Ferromagnetism

We end our discussion of magnetostatics in matter by examining ferromagnetism.

A material is said to be ferromagnetic if it retains its magnetization even after the applied field is turned off, forming permanent magnets.

The explanation of this effect is purely quantum mechanical, so we shall only try to provide a qualitative explanation.

Some materials have localized regions of magnetization, called domains. Although each domain does produce a magnetic field, in its natural state the overall the dipole moments of the domains are oriented randomly, cancelling each other out.

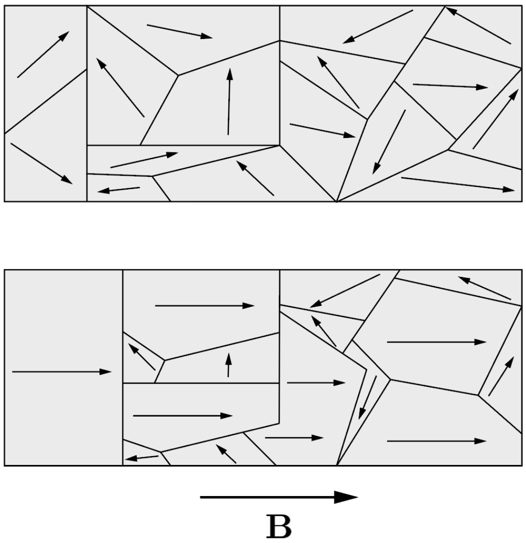


Figure 5.3. Domain resizing and alignment in ferromagnets

When a magnetic field is applied, of course one expects the domains in the ferromagnet to gradually get aligned, like in paramagnets [1]. There is an additional factor in ferromagnets however, which occurs at the boundary between domains. Here, we have two competing regions of dipoles, all the dipoles on one side are oriented in some direction, and all dipoles on the other are oriented in another direction. The domain pointing most along the applied field will win this "tug of war", causing a resizing in the magnetic domains. Eventually, after a strong enough field is applied, one domain takes over entirely, saturating the object.

The resizing of domains is not reversible, and even after the field is reduced some of the magnetization will remain. Most domains will therefore point in the direction of the field, forming a permanent magnet.

To remove this magnetization, one would need to apply a magnetic field antiparallel to the original field. At some point then the magnetization will reduce to zero. If this is overdone

[1]this means that ferromagnets mostly occurs with atoms with unpaired electrons

however, one may get the opposite effect, that is a magnetization to the opposite of the original magnetic field's direction.

Notice that the magnetization does not merely depend on the applied current, but also on the history of the material. For example, there are three points where the applied field was null, each with different values of magnetization.

We can describe the behaviour of ferromagnets using a magnetic hysteresis loop:

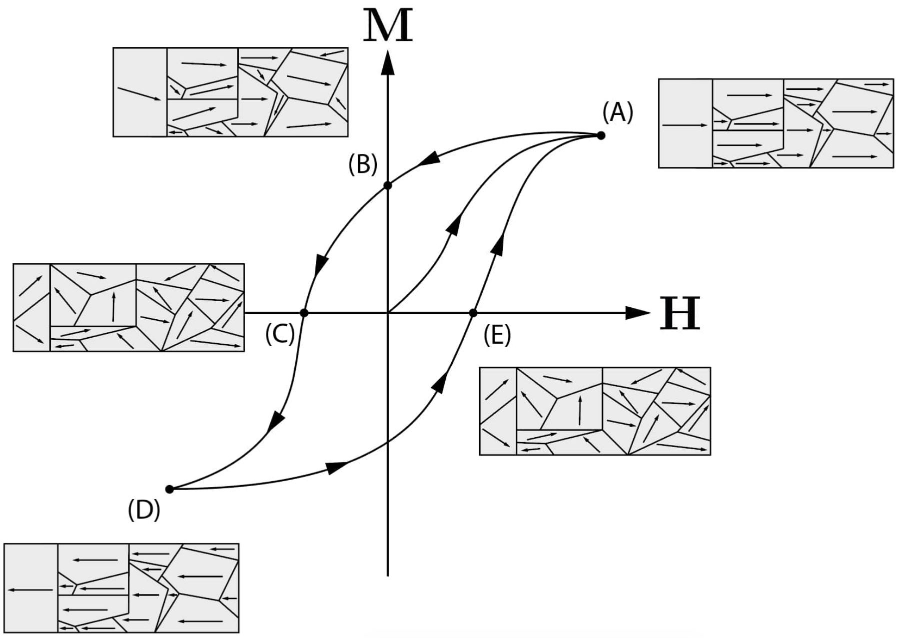


Figure 5.4. Typical magnetic hysteresis curve

Part III

Electromagnetism

6 Motion in electromagnetic fields

The goal of this chapter will be to model the motion of particles under different electromagnetic fields.

6.1 Motion in uniform fields

Let us firstly consider a particle of mass m, charge q, under the influence of uniform electromagnetic fields, independent of time and position.

If $\mathbf{E}$ is constant and $\mathbf{B} = \mathbf{0}$ then:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\mathbf{E} \implies \frac{d\mathbf{v}}{dt} = \frac{q\mathbf{E}}{m} \tag{6.1.1}$$

This is analogous to the motion under a uniform gravitational field $\mathbf{g}$, and the solution is therefore:

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{q\mathbf{E}}{2m}t^2 \tag{6.1.2}$$

If instead $\mathbf{B}$ is constant and $\mathbf{B} = \mathbf{0}$, then the Lorentz force law reads:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \tag{6.1.3}$$

We can rotate the coordinate axes so that $\mathbf{B} = B\hat{\mathbf{z}}$, in which case we get the following coupling:

$$\frac{dv_x}{dt} = \frac{qB}{m}v_y \tag{6.1.4}$$

$$\frac{dv_y}{dt} = -\frac{qB}{m}v_x \tag{6.1.5}$$

$$\frac{dv_z}{dt} = 0 \tag{6.1.6}$$

Differentiating the first two equations with respect to time:

$$\frac{d^2v_x}{dt^2} = \frac{qB}{m}\Big(-\frac{qB}{m}v_x\Big) \implies \frac{d^2v_x}{dt^2} + \frac{qB}{m}v_x = 0 \tag{6.1.7}$$

$$\frac{d^2v_y}{dt^2} = -\frac{qB}{m}\Big(\frac{qB}{m}v_y\Big) \implies \frac{d^2v_y}{dt^2} + \frac{qB}{m}v_y = 0 \tag{6.1.8}$$

$$\tag{6.1.9}$$

we get back the simple harmonic oscillator equations. These have the typical sinusoidal solutions:

$$v_x = v_\perp \sin\left(\frac{qB}{m}t + \phi_0\right) \tag{6.1.10}$$

$$v_y = v_\perp \cos\left(\frac{qB}{m}t + \phi_0\right) \tag{6.1.11}$$

where $v_\perp = v_x^2 + v_y^2$ is the speed in the plane perpendicular to the magnetic field. Since $v_z = v_\parallel$ and $v^2 = v_x^2 + v_y^2 + v_z^2$ are constant , we must have that $v_\perp$ is also constant. Consequently:

$$\mathbf{r} = \left[x_0 - \frac{mv_\perp}{qB}\cos\left(\frac{qB}{m}t + \phi_0\right)\right]\hat{\mathbf{x}} + \left[y_0 + \frac{mv_\perp}{qB}\sin\left(\frac{qB}{m}t + \phi_0\right)\right]\hat{\mathbf{y}} + (z_0 + v_\parallel t)\hat{\mathbf{z}} \tag{6.1.12}$$

We see that from the point of view of the z-axis, the particle undergoes uniform circular motion with frequency:

$$\omega_c = \frac{|q|B}{m} \tag{6.1.13}$$

known as the **cyclotron frequency**. The radius of the circular path is:

$$r_c = \frac{mc_\perp}{qB} \tag{6.1.14}$$

We could have also derived this result using Newtonian mechanics:

$$F_r = qv_\perp B = \frac{mv_\perp^2}{r} \implies r = \frac{mv_\perp}{qB} \tag{6.1.15}$$

where F_r is the radial force component.

The most important application of our results is in the cyclotron particle accelerator. It is made of two hollow semi-circle dees through which a uniform magnetic field **B** perpendicular to the plane of the dees is present. A voltage is applied between the two dees.

We see that the particle injected in the dees will travel in circular paths, and will be accelerated by the electric field in the gap. To make sure that the electric field points in the right direction to accelerate the particle, we need to reverse the potential at the cyclotron frequency ω_c.

We have that the outer radius of the cyclotron and the magnetic field determine with how much energy the particles will exit the accelerator, whereas the potential determines how quickly this will happen.

Now suppose we apply uniform magnetic and electric fields parallel to each other. The equation of motion of such a particle will be:

$$\mathbf{v} = v_\perp \sin\left(\frac{qB}{m}t + \phi_0\right)\hat{\mathbf{x}} + v_\perp \cos\left(\frac{qB}{m}t + \phi_0\right)\hat{\mathbf{y}} + \left(v_{0\parallel} + \frac{qE}{m}t\right)\hat{\mathbf{z}} \tag{6.1.16}$$

This set up is particularly useful when trying to collimate particle beams. Indeed, using a uniform electric field to accelerated electrons will not produce a narrow beam. As the particles are accelerated more and more, the beam will continue to diverge. The average speed perpendicular to the direction of the electric field will be $v_\perp \sim \sqrt{\frac{2k_BT}{m}}$ (see Thermal physics volume). Instead the average component of the velocity parallel to the field is $v_\parallel = \frac{2eV}{m}$. For typical values of $V = 2.5kV$ for an electron at $2700K$, we will find that $v_\parallel : v_\perp = 100 : 1$. As the particle travels $50m$ in the electric field's direction, the beam radius will be approximately $50cm$.

To reduce this divergence on can apply a magnetic field parallel to the beam, which will cause the particles to move in a helical trajectory. Every multiple of $T_c = \frac{2\pi}{\omega_c}$, the particles return to the axis of the electric field, and are therefore collimated. Moreover, even at different distances from the ones travelled in time T_c the beam radius is restricted to the cyclotron radius $r_c = \frac{mv_\perp}{eB}$ which for the situation described previously is ten times smaller.

6.2 Motion in non-uniform fields

We begin by considering $\mathbf{B} = B\hat{\phi}$.

7

Quasistatic Approximations

7.1 Ohm's Law

Generally, the current density through a conductor is proportional to the force per unit charge on the charge carriers:

$$\mathbf{J} = \sigma \mathbf{f} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{7.1.1}$$

where σ is the conductivity. Conductors satisfying such a relation are called **Ohmic conductors**. Using the Lorenz Force Law with $\mathbf{v} \approx 0$ one then finds that:

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}. \tag{7.1.2}$$

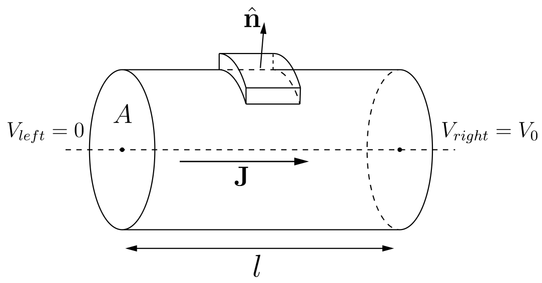


Figure 7.1. Cylindrical Resistor's Boundary Conditions

For an archetypal resistor modelled as a cylinder of cross sectional area A and length l with a potential difference V_0 between its ends, then there will be a uniform electric field in the conductor. Indeed, we have the following boundary conditions:

$$V_{left} = 0, \quad V_{right} = V_0, \quad \mathbf{J} \cdot \hat{\mathbf{n}} = 0 \implies \mathbf{E} \cdot \hat{\mathbf{n}} = 0 \implies \frac{\partial V}{\partial n} = 0 \tag{7.1.3}$$

where the latter comes from the fact that charge doesn't leak out of the resistor. We now read off a solution:

$$V(z) = \frac{V_0 z}{l} \implies \mathbf{E} = -\frac{V_0}{l}\hat{\mathbf{n}} = 0 \quad (z \text{ is the distance along the axis}) \tag{7.1.4}$$

which is the only solution by the uniqueness theorem. Given that the electric field is constant, hence:

$$I = JA = \sigma EA = \frac{\sigma A}{l}V \implies V = \frac{\rho l}{A}I \tag{7.1.5}$$

where $\rho = \frac{1}{\sigma}$ is the resistivity (how much the material resists the flow of current). More generally, we find that the potential difference across a resistor is proportional to the current flowing through it:

$$\boxed{V = IR} \tag{7.1.6}$$

where R is called the resistance. Since the potential difference V is a measure of work done per unit charge, and current I is the charge flowing per unit time, it follows that their product gives the power delivered by the source to the resistor:

$$\boxed{P = IV = I^2R} \tag{7.1.7}$$

Definition: Ohm's law

Ohmic conductors are such that:

$$\mathbf{J} = \sigma\mathbf{f} \approx \sigma\mathbf{E} \tag{7.1.8}$$

or more commonly:

$$V = IR \tag{7.1.9}$$

where R is the **resistance** of the conductor, V the potential difference at its ends, and I is the current through it.

7.2 Electromotive Force

However, we have an apparent contradiction, since if a constant electric field generates a current, then the particles should be accelerating, causing an increase in current. Yet, if we attach the a lamp to a circuit, its brightness will remain constant. Furthermore, one would expect the current to be greater near the battery, and almost none through the lamp. So why is a steady current generated?

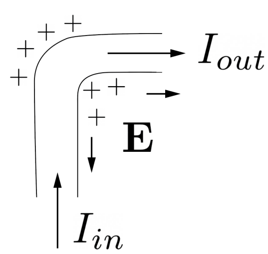


Figure 7.2. Stable equilibrium of steady currents

In reality, there are other forces at play in addition to the Lorenz Force, and these help "even out" the current density. Firstly, one must consider the collisions between the various charge carriers, which definitely decrease the average speed of the charges overall. Secondly, if the current were not steady, then charge would pile up somewhere, and this would create an electric field in the direction that opposes the accumulation. Thus, the current is self-correcting, immediately killing off any fluctuation in the current density.

This is also why there is no decrease in current through a resistor/conductor, but rather a voltage drop.

Take a piece of wire in the circuit, and suppose the current in is greater than the current out $I_{in} > I_{out}$ so that positive charges pile up at the edges of the wire. This pile of charges produces an electric field which reinforces I_out and weakens I_in.

It follows from this discussion that the force "pushing" the charges through a circuit is not simply due to the source (e.g. battery) but also electrostatic in nature:

$$\mathbf{f} = \underbrace{\mathbf{f}_s}_{\text{source}} + \mathbf{E} \tag{7.2.1}$$

and we can thus define the **electromotive force to be**:

$$\boxed{\varepsilon = \oint_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{l} = \oint_{\mathcal{C}} \mathbf{f}_s \cdot d\mathbf{l}} \tag{7.2.2}$$

since $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for electrostatic fields. Now, for a perfect source of electromotive force, we have perfect conductivity, and thus taking $\sigma \to \infty$ gives $\mathbf{f} = \mathbf{f}_s + \mathbf{E} \approx 0$. The voltage across the terminals of the source is then:

$$V_{\text{terminals}} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} \tag{7.2.3}$$

and since $\mathbf{f}_s = 0$ outside the source, we can extend the line integral around the entire circuit:

$$V_{\text{terminals}} = \oint_{\mathcal{C}} \mathbf{f}_s \cdot d\mathbf{l} = \varepsilon \tag{7.2.4}$$

Therefore, the source therefore maintains a constant voltage across the terminals of a circuit, and the resultant electric field then drives the current.

Electromotive force

The **electromotive force** supplied by a source is defined to be:

$$\varepsilon = \oint_{\mathcal{C}} \mathbf{f}_s \cdot d\mathbf{l} \tag{7.2.5}$$

where $\mathbf{f}_s$ is the force per unit charge.

A source doesn't always necessarily need to have an emf. Indeed, motional electromotive forces arise when a conductor is moved through a magnetic field. In such case, we have that the induced emf will be proportional to the instantaneous rate of change in magnetic flux through the circuit loop:

$$\boxed{\varepsilon = -\frac{d\Phi}{dt}}. \tag{7.2.6}$$

Consider the following loop:

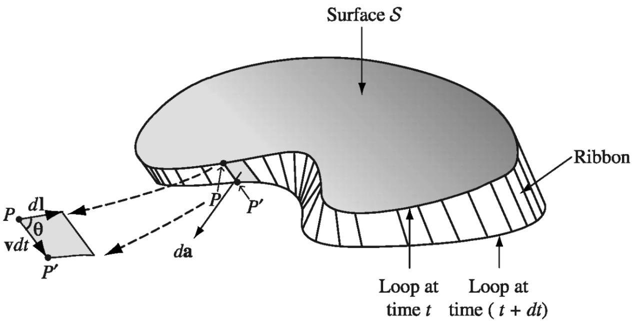


Figure 7.3. Circuit at time t and $t + \Delta t$ producing a ribbon

with boundary $\partial S(t)$ at time t and subsequently $\partial S(t + \Delta t)$ at time $t + \Delta t$, enclosing the open surfaces $S(t)$ and $S(t + \Delta t)$ respectively. Then, $S(t + \Delta t) \setminus S(t)$ will form a "ribbon":

$$d\Phi = \Phi(t + \Delta t) - \Phi(t) = \Phi_{ribbon} = \int_{ribbon} \mathbf{B} \cdot d\mathbf{a} \tag{7.2.7}$$

If the circuit is moved with velocity $\mathbf{v}$ and the charge carriers move with velocity $\mathbf{u}$ relative to the circuit, then a particle at P will overall be moving with velocity $\mathbf{w} = \mathbf{v} + \mathbf{u}$. Then:

$$d\mathbf{a} = (\mathbf{v} \times d\mathbf{l})dt \implies d\Phi = \int_{ribbon} \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l})dt \tag{7.2.8}$$

Note that $\mathbf{v} \times d\mathbf{l} = \mathbf{w} \times d\mathbf{l} - \cancelto{0}{\mathbf{u} \times d\mathbf{l}} = \mathbf{w} \times d\mathbf{l}$ so:

$$\frac{d\Phi}{dt} = \int_{ribbon} \mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}) = -\int_{ribbon} d\mathbf{l} \cdot \underbrace{(\mathbf{w} \times \mathbf{B})}_{\mathbf{f}_{mag}} \tag{7.2.9}$$

Therefore, we find that:

Motional electromotive force

The **motional emf** produced when moving a conductor through a magnetic field $\mathbf{B}$ is given by:

$$\varepsilon = -\frac{d\Phi}{dt} \tag{7.2.10}$$

where Φ is the magnetic flux through the conductor.

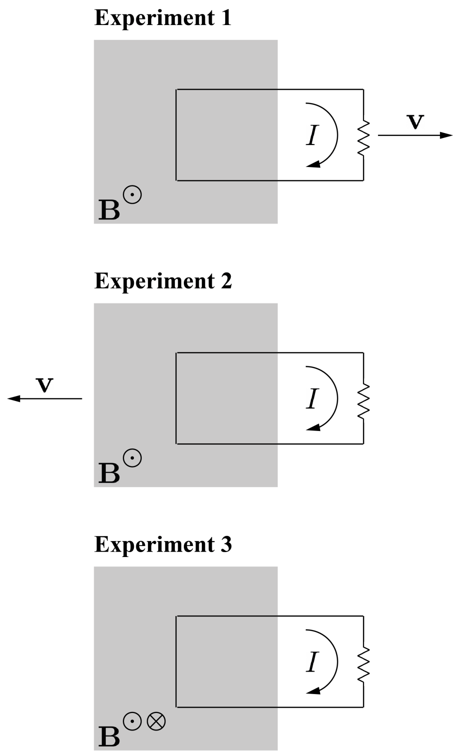
One question remains: suppose we take a loop of wire connected to a resistor, and move it through a magnetic field. The motional emf will induce a current through the wire, turning the light bulb on, and heating it. But where does this enery come from?

Certainly not from the magnetic field, since they do no work. The work is instead done by whomever is pulling the wire through the magnetic field.

7.3 Electromagnetic Induction

Michael Faraday performed three experiments in 1831 which led to the discovery of induction:

1. **Experiment 1:** he pulled a loop of wire to the right through a uniform magnetic field, a current flowed.
2. **Experiment 2:** he moved the magnet to the left, holding the loop still, a current flowed.
3. **Experiment 3:** with both the loop and magnet at rest, he changed the strength and direction of the field, a current flowed.



While the first experiment was just a demonstration of motional emf (Lorentz force), the other two clearly show that *a changing magnetic field induces an electric field.* It was empirically found that the emf due to the induced electric field was equal to the motional emf:

$$\varepsilon = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \qquad (7.3.1)$$

as would be expected by relativistic considerations. Using Stoke's theorem, we arrive at Faraday's Law:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \qquad (7.3.2)$$

To determine the signs and direction of the induced currents, one can use Lenz's Law:

$$\boxed{\text{Nature abhors any change in magnetic flux}} \qquad (7.3.3)$$

Therefore, the induced current will flow in a direction such that it tries to kill off the initial change in flux. Consider for example the famous demonstration in which a solenoidal coil is wound around a metal core and a current is allowed to flow through it. Then, if we place a conductive metal ring on the metal core it will jump upwards. This is because before turning the current on, there was no flux through the ring, whereas when the current starts flowing through the coil, the resultant magnetic field will cause a change in magnetic flux, inducing a current through the ring. This current will run opposite to the current in the solenoid so as to kill off the change in flux, and since opposite currents repel the ring jumps.

We can see Faraday's Law as simply the time-dependent version of Ampere's Law. Indeed, for a Faraday field with no net charge, we find:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

so we simply replace $\mu_0 \mathbf{J}$ with $-\frac{\partial \mathbf{B}}{\partial t}$ in magnetostatics. For example, Ampere's Law $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$ would turn into Faraday's law $\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{da}$. Similarly, the Biot-Savart equivalent for time-dependent currents becomes:

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi} \frac{\partial}{\partial t} \left(\int \frac{\mathbf{B}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau \right) \tag{7.3.4}$$

Due to this mathematical coincidence, it is evident that if we wish to find the direction of the electric field, it suffices to use the cork-screw rule with the thumb pointing opposite to the direction of the change in the magnetic field. Of course, the more rigorous symmetrical arguments may also be used.

Note that we are using our tools from static fields in order to study time-dependent fields, which may sound quite sketchy. Luckily, it turns out that the results we have derived are approximately correct, and that the error is negligible unless the field fluctuations are very sudden (which we shall study in the radiation chapter). Such slowly varying fields are called **quasistatic fields**.

Electromagnetic induction

Nature abhors any change in magnetic flux, and such a change induces an electric field as stated in Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi} \frac{\partial}{\partial t} \left(\int \frac{\mathbf{B}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau \right) \tag{7.3.5}$$

7.4 Inductance and energy in magnetic field

Let us consider two current-carrying loops, as shown below.

Note that the magnetic field produced by loop 1 will produce a net flux through loop 2 given by:

$$\Phi_2 = \int_{\mathcal{S}_2} \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int_{\mathcal{C}_2} \mathbf{A}_1 \cdot d\mathbf{l}_2 \tag{7.4.1}$$

and since:

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{\mathcal{C}_1} \frac{d\mathbf{l}_1}{|\mathbf{r} - \mathbf{r}'|} \tag{7.4.2}$$

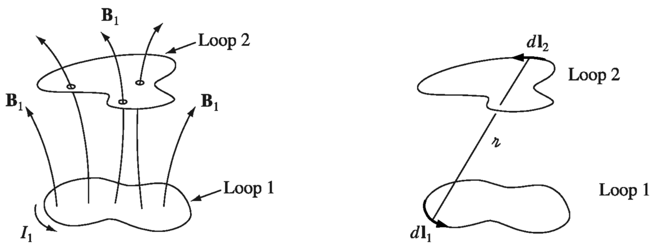


Figure 7.4. Mutual Inductance between two loops

we find that:

$$\Phi_2 = I_1 \underbrace{\frac{\mu_0}{4\pi} \oint_{\mathcal{C}_2} \oint_{\mathcal{C}_1} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{r} - \mathbf{r}'|}}_{\text{mutual inductance}} = I_1 M_{21} \tag{7.4.3}$$

so the flux through loop 2 is directly proportional to the current in loop 1. The constant proportionality is called *mutual inductance*, denoted M_{21}. Note that:

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_2} \oint_{\mathcal{C}_1} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{d\mathbf{l}_2 \cdot d\mathbf{l}_1}{|\mathbf{r} - \mathbf{r}'|^3} = M_{12} \tag{7.4.4}$$

so if $I_1 = I_2 = I$, no matter what shape and size of the two loops, the flux through loop 1 due to loop 2 carrying current I will be the same as the flux through loop 2 due to loop 1 carrying current I.

If we let I_1 vary with time, then an emf, ε_2, will be induced in loop 2, given by:

$$\varepsilon_2 = -\frac{d\Phi}{dt} = -M\frac{dI_1}{dt} \tag{7.4.5}$$

The changing current, surprisingly, will also induce an emf in loop 1 itself. If we define the flux through loop 1 to be:

$$\boxed{\Phi = LI} \tag{7.4.6}$$

where L is the *self-inductance* of the loop, then:

$$\varepsilon = -L\frac{dI}{dt} \tag{7.4.7}$$

This emf is called the back emf, because by Lenz's law it acts against the change in flux, and thus current. Therefore, whenever one tries to change the current through a loop, there will be an opposing reaction force due to the back emf.

Inductance

The mutual inductance M between two loops of wire carrying currents I_1 and I_2 is such that it satisfies:

$$\Phi_2 = MI_1 \quad \text{and} \quad \Phi_1 = MI_2 \tag{7.4.8}$$

so that the emf induced in loop 2 due to a change in I_1 is:

$$\varepsilon_2 = -L\frac{dI_1}{dt} \tag{7.4.9}$$

The self inductance L of a loop of wire carrying current I is a quantity satisfying:

$$\Phi = LI \tag{7.4.10}$$

so that the back emf, the work per unit charge experienced when trying to alter the current I is:

$$\varepsilon = -L\frac{dI}{dt} \tag{7.4.11}$$

Simply put:

$$\text{Resistance is the opposition to } \textit{flow} \text{ of current.} \tag{7.4.12a}$$
$$\text{Inductance is the opposition to } \textit{change} \text{ of current.} \tag{7.4.12b}$$

An implication of this astounding result is that, whenever we try to start a current flowing in a circuit, a certain amount of work must be done against the back emf, which is stored in the magnetic field of the current until it is turned off. To calculate this energy:

$$\frac{dW}{dt} = -\varepsilon I = LI\frac{dI}{dt} \tag{7.4.13}$$

which upon integration yields:

$$\boxed{W = \frac{1}{2}LI^2} \tag{7.4.14}$$

Also, notice that since $\Phi = LI = \oint \mathbf{A} \cdot d\mathbf{l}$, the previous expression allows us to write:

$$\begin{aligned}
W &= \frac{1}{2}I\oint \mathbf{A} \cdot d\mathbf{l} \\
&= \frac{1}{2}\oint (\mathbf{A} \cdot \mathbf{I})dl \\
\implies W &= \frac{1}{2}\int (\mathbf{A} \cdot \mathbf{J})d\tau
\end{aligned}$$

where the latter is a generalization to volume currents.

By Ampere's Law, $\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} \implies \mathbf{A}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{A}\cdot(\nabla\times\mathbf{B})$, so:

$$\begin{aligned} W &= \frac{1}{2\mu_0}\int_{\mathcal{V}} B^2 - \nabla\cdot(\mathbf{A}\times\mathbf{B})d\tau \\ &= \frac{1}{2\mu_0}\left(\int_{\mathcal{V}} B^2 d\tau - \oint_{\mathcal{S}} \mathbf{A}\times\mathbf{B}\cdot d\mathbf{a}\right) \\ &= \frac{1}{2\mu_0}\int_{\mathbb{R}^3} B^2 d\tau \end{aligned}$$

as we let $\mathcal{V}\to\mathbb{R}^3$. So finally:

$$\boxed{W = \frac{1}{2\mu_0}\int_{\mathbb{R}^3} B^2 d\tau} \tag{7.4.15}$$

Note the remarking similarities with the electrostatic equivalents:

$$W = \frac{1}{2}CV^2 = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} E^2 d\tau \tag{7.4.16}$$

Magnetic field energy

The energy stored in a magnetic field $\mathbf{B}$ with inductance L and current I is:

$$W = \frac{1}{2}LI^2 = \frac{1}{2\mu_0}\int_{\mathbb{R}^3} B^2 d\tau \tag{7.4.17}$$

7.5 Maxwell's Displacement Current

One issue that must be addressed is the mathematical inconsistency in Ampere's Law:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{7.5.1}$$

Indeed, taking the divergence yields:

$$\nabla\cdot(\nabla\times\mathbf{B}) = 0, \quad \text{and } \nabla\cdot(\mu_0\mathbf{J}) = \mu_0\nabla\cdot\mathbf{J} \tag{7.5.2}$$

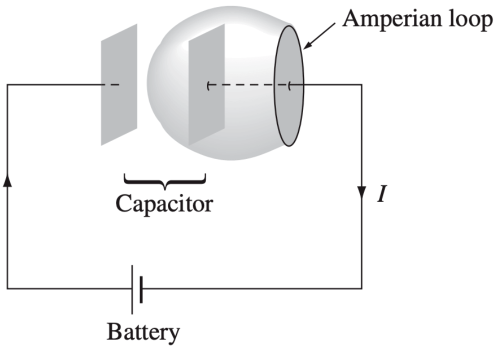
where the RHS is only zero for time-independent currents (steady currents), where charge does not accumulate.

Alternatively, consider the following circuit where a capacitor is being charged by a battery:

We take a circular Amperian loop through which a current I flows, so that Ampere's law reads:

$$\oint_{\mathcal{C}} \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc} \tag{7.5.3}$$

where I_{enc} is the current enclosed by any open surface $\mathcal{S}$ enclosed by $\mathcal{C}$. However, I_{enc} is



ill-defined, since it changes according to the surface chosen. If we chose the standard disk enclosed by the loop, we find that $I_{enc} = I$, whereas if we choose a balloon enclosing one of the plates, then $I_{enc} = 0$. Clearly, when charge starts accumulating somewhere, Ampere's law breaks down.

So how do we make $\nabla \cdot \mathbf{J} = 0$? Note that the continuity equation can be written as:

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}\overbrace{(\epsilon_0 \nabla \cdot \mathbf{E})}^{\rho} = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \tag{7.5.4}$$

So if we subtract this last term to $\mathbf{J}$ (in order to make $\nabla \cdot (\nabla \times \mathbf{B}) = 0$) we reach the Maxwell-Ampere Law:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)} \tag{7.5.5}$$

The extra term we added is called the displacement current:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{7.5.6}$$

Definition: MAXWELL'S DISPLACEMENT CURRENT

The Maxwell-Ampere law reads:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \tag{7.5.7}$$

where

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{7.5.8}$$

is called the **Maxwell displacement current**, and eextends Ampere's law to non-steady currents where charges accumulate.

In matter it is easy to generalize the argument. Recall that here Ampere's law reads:

$$\nabla \times \mathbf{H} = \mathbf{J}_f \tag{7.5.9}$$

so taking the divergence on both sides we get that $\nabla \cdot \mathbf{J}_f = 0$. This is not always true, so we must modify this law. The continuity equation reads:

$$\nabla \times \mathbf{H} = \nabla \cdot \mathbf{J}_f = -\frac{\partial}{\partial t}\overbrace{(\nabla \cdot \mathbf{D})}^{\rho_f} = -\nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t}\right) \tag{7.5.10}$$

so that:

$$\mathbf{J}_f \to \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{7.5.11}$$

and thus:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}} \tag{7.5.12}$$

In LIH materials where $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$ and $\mathbf{D} = \epsilon \mathbf{E}$ we find that:

$$\nabla \times \mathbf{B} = \mu \mathbf{J}_f + \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{7.5.13}$$

7.6 Boundary Conditions

We summarise the Maxwell equations in their completeness below:

$$\boxed{\begin{aligned} &\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && (7.6.1) \\ &\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} && (7.6.2) \end{aligned}}$$

in differential form and

$$\boxed{\begin{aligned} &\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{enc}^f \qquad \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0 && (7.6.3) \\ &\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\left(\int_{\mathcal{S}\setminus\mathcal{C}} \mathbf{B} \cdot d\mathbf{a}\right) \qquad \oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{enc}^f + \frac{d}{dt}\left(\int_{\mathcal{S}\setminus\mathcal{C}} \mathbf{D} \cdot d\mathbf{a}\right) && (7.6.4) \end{aligned}}$$

in integral form.

Applying the first to a wafer thin pillbox:

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f a \tag{7.6.5}$$

and therefore

$$D_1^\perp - D_2^\perp = \sigma_f \tag{7.6.6}$$

Similarly, the second equation on the top gives:

$$B_1^{\perp} - B_2^{\perp} = 0 \tag{7.6.7}$$

Applying the third to a thin amperian loop so that $\Phi \to 0$ then:

$$(\mathbf{E}_1 - \mathbf{E}_2) \cdot \mathbf{l} = -\frac{d\Phi}{dt} = 0 \tag{7.6.8}$$

so that:

$$\mathbf{E}_1^{||} - \mathbf{E}_2^{||} = \mathbf{0} \tag{7.6.9}$$

Similarly, for the fourth equation:

$$(\mathbf{H}_1 - \mathbf{H}_2) \cdot \mathbf{l} = I_{enc} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l} \tag{7.6.10}$$

hence we find that:

$$\mathbf{H}_1^{||} - \mathbf{H}_2^{||} = \mathbf{K}_f \times \hat{\mathbf{n}} \tag{7.6.11}$$

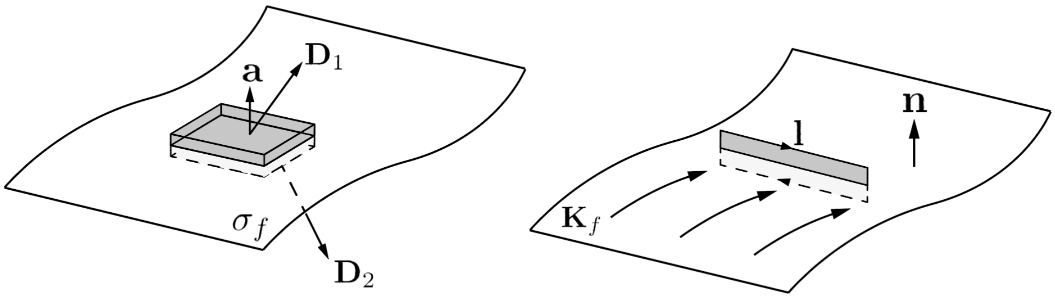


Figure 7.5. Boundary conditions for electromagnetic fields

Boundary conditions for electromagnetic fields

In summary, for a surface with fields $\mathbf{E}_1, \mathbf{B}_1$ above and $\mathbf{E}_2, \mathbf{B}_2$ below:

$$\boxed{\begin{array}{ll} D_1^{\perp} - D_2^{\perp} = \sigma_f & \mathbf{E}_1^{||} - \mathbf{E}_2^{||} = 0 \\ B_1^{\perp} - B_2^{\perp} = 0 & \mathbf{H}_1^{||} - \mathbf{H}_2^{||} = \mathbf{K}_f \times \hat{\mathbf{n}} \end{array}} \qquad \begin{array}{r} (7.6.12) \\ (7.6.13) \end{array}$$

For LIH materials the boundary conditions become:

$$\boxed{\begin{array}{ll} \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f & \mathbf{E}_1^{||} - \mathbf{E}_2^{||} = 0 \\ B_1^{\perp} - B_2^{\perp} = 0 & \dfrac{1}{\mu_1}\mathbf{B}_1^{||} - \dfrac{1}{\mu_2}\mathbf{B}_2^{||} = \mathbf{K}_f \times \hat{\mathbf{n}} \end{array}} \qquad \begin{array}{r} (7.6.14) \\ (7.6.15) \end{array}$$

Superconductivity

8

8.1 History of superconductivity

The discovery of superconductivity was made by Heike Kamerlingh Onnes in 1911 when studying the behaviour of metals at very low temperatures. While measuring the electrical resistance of a column of mercury, he observed that the resistance fell significantly from around 0.1Ω at 4.3 K to $3 \times 10^{-6}\Omega$ at 4.1 K.

We say that below $4.1K$, known as the **critical temperature** T_c, mercury behaves as a **superconductor**, a conductor which does not resist to a flow of a steady current. It was later discovered that this phenomenon of superconductivity is not special to mercury, but actually occurs in several metals, and even non-metals when under extreme pressure.

The next major breakthrough would come in 1933, when Walter Meissner and Robert Oschenfeld observed that superconductors are not only perfect conductors, but also perfect diamagnets, in other words they will expel any applied magnetic field (as long as it is below a critical field strength). Meissner and Ocshenfeld tried to provide a semi-classical description, later followed by the London brothers in 1935.

The most accurate model for superconductors, knwon as the BCS theory, would be developed only twenty years later by John Bardeen, Leon Cooper and John Schrieffer. Two years later, Alexei Abrikosov predicted a new type of superconductor for which magnetic fields would be expelled if under a critical field strength, but would be penetrable by the magnetic field only in certain regions if this field strength threshold was surpassed.

In the decades to follow, more and more materials with increasing critical temperatures were discovered. The highest critical temperature ever achieved was 135 K, definitely not room temperature. Nevertheless, it is much higher than the boiling point of liquid nitrogen, which can now be used as a coolant for superconductors.

8.2 Qualitative analysis of superconductors

The main characteristic of superconductors is the reduction of its electrical resistance to zero below a temperature T_c known as the critical temperature.

This property has been tested several times, for example by letting current flow in a superconductive ring and observe whether the produced magnetic field decays over time in absence of a source of emf.

For example, Quinn and Ittner performed an experiment in 1962 to test the persistency of superconductive currents. Two thin films of superconducting lead were separated by a layer of insulating silicon dioxide.

Assuming the system has self-inductance L, a current I_0 flowing initially in absence of an emf source, and a small resistance R, we find that the current after some time t satisfies:

$$0 = L\frac{dI}{dt} + IR \implies I(t) = I_0 e^{-Rt/L} \tag{8.2.1}$$

In their experiment, the inductance of the tube was measured to be $L = 1.4 \times 10^{-13}$ H, and no change in magnetic moment was detected after 7 hours, with a measurement error of 2%, meaning that at least 98% of the original value of the current was still flowing. It follows that the maximum possible resistance:

$$I(7\text{ hours}) = 0.98 I_0 \implies R_{max} = -\frac{L}{t}\ln 0.98 \approx 1.12 \times 10^{-19}\Omega \tag{8.2.2}$$

Consequently, the maximum possible resistance is approximately $1.12 \times 10^{-19}\Omega$, an extremely small quantity.

From this result we can easily find the value of the conductivity:

$$R = \frac{l}{\sigma A} \implies \sigma = \frac{l}{R \cdot A} = \frac{6.4\text{ mm}}{1.12 \times 10^{-19}\Omega \cdot 17\text{ mm} \cdot 1.2 \times 10^{-3}\text{ mm}} \tag{8.2.3}$$

$$\approx 2.80 \times 10^{24}\Omega^{-1}\text{ m}^{-1} \tag{8.2.4}$$

The conductivity of lead at $273K$ is $5.2 \times 10^6\Omega^{-1}\text{ m}^{-1}$, which is almost 18 orders of magnitude smaller than that measured in the superconductive experiment.

A second interesting property is that superconducting rings have a constant magnetic flux.

Indeed, suppose we have a ring of metal above the critical temperature, inserted in a uniform magnetic field $\mathbf{B}_0$ perpendicular to the plane of the ring. Then, the initial magnetic flux will be $B_0 A$. We then lower the temperature of the ring below the critical temperature, so that it becomes superconductive.

If we now change the magnetic field, there will be an induced emf in the ring and thus an induced current I related by Faraday's law:

$$V_{emf} = -\frac{d\Phi}{dt} = -A\frac{dB}{dt} = L\frac{dI}{dt} \tag{8.2.5}$$

where $R = 0$. Then:

$$\frac{d}{dt}(LI + BA) = 0 \tag{8.2.6}$$

This term is exactly the total magnetic flux through the ring. Indeed the first term is the flux due to the ring's own current-induced magnetic field, whereas the second term is the flux due to the applied field.

An important application of this property of superconductors is in circuits where it is important for the magnetic field to remain constant to a high degree of precision. A superconducting solenoid above critical temperature is connected to a power supply. This produces a magnetic field, which we stabilize by cooling the superconducting solenoid below critical temperature, thus fixing the flux and hence the magnetic field in the solenoid.

Superconductors also expel any applied magnetic field by acting as perfect diamagnets, a phenomenon known as the **Meissner effect**.

A perfect conductor, as was seen earlier, is one with no resistance and thus with constant magnetic flux. This means that $\frac{\partial \mathbf{B}}{\partial t} = 0$ for the field $\mathbf{B}$ inside the conductor. Consequently, if we cool the perfect conductor first, and then apply the magnetic field it will get screened completely. However, if we apply the magnetic field first, and then cool the perfect conductor, then the field cannot be expelled since the magnetic flux must remain constant. Even if we remove the applied field the magnetic field inside the perfect conductor will remain constant (the change in the field will induce currents to counterbalance the change in flux).

Compare this to the case of superconductors, where in either case the magnetic field in the interior of the superconductor gets screened completely.

Interestingly, superconductivity gets destroyed above a certain magnetic field strength, known as the **critical magnetic field strength** B_c. The temperature dependence of B_c is parabolic:

$$B_c(T) = B_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right] \tag{8.2.7}$$

8.3 The London equations

We model the free electrons in a superconductor as two fluids, one consisting of normal electrons with number density n_N, and another consisting of superconducting electrons with number density n_S. If the mean time between normal electron collisions is τ, then the mean drift velocity of the normal electrons accelerated by an electric field $\mathbf{E}$ is:

$$\langle v \rangle_n = -\frac{e\tau\mathbf{E}}{m} \tag{8.3.1}$$

Therefore:

$$\mathbf{J}_n = -n_N e \langle \mathbf{v}_n \rangle = \frac{n_N e^2 \tau}{m}\mathbf{E} \tag{8.3.2}$$

The superconducting electrons do not collide, they encounter no resistance to flow so we may model this second fluid as dissipationless

$$m\frac{d\mathbf{v}_s}{dt} = -e\mathbf{E} \tag{8.3.3}$$

We therefore find that:

$$\boxed{\frac{\partial \mathbf{J}_s}{\partial t} = \frac{n_s e^2}{m}\mathbf{E}} \tag{8.3.4}$$

known as the **first London equation**.

Note that for a steady current, $\frac{\partial \mathbf{J}_s}{\partial t} = 0$ or equivalently $\mathbf{E} = 0$. Hence, the only contribution to the steady current in the superconductor is due to the superconducting electrons, and not the normal electrons.

We model the superconductor as either weakly diamagnetic or weakly paramagnetic, implying that $\mu \approx 1$ and thus $\mathbf{H} \approx \frac{\mathbf{B}}{\mu_0}$. We can then take the curl of (8.3.4) and use Faraday's law:

$$\nabla \times \left(\frac{\partial \mathbf{J}_s}{\partial t}\right) = \frac{n_s e^2}{m}\nabla \times \mathbf{E} = -\frac{n_s e^2}{m}\frac{\partial \mathbf{B}}{\partial t} \tag{8.3.5}$$

and integrating both sides:

$$\nabla \times \mathbf{J}_s = -\frac{n_s e^2}{m}\mathbf{B} + \mathbf{c}(\mathbf{r}) \tag{8.3.6}$$

We can now take the curl of Ampere's law ignoring the displacement term since we are interested in steady currents to get:

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \nabla \times \mathbf{J}_s = \mu_0\Big(-\frac{n_s e^2}{m}\mathbf{E}\mathbf{B} + \nabla \times \mathbf{c}\Big) \tag{8.3.7}$$

and using the vector identity $\nabla \times (\nabla \times \mathbf{B}) = \nabla\cancelto{0}{(\nabla \cdot \mathbf{B})} - \nabla^2\mathbf{B}$ we find that:

$$\nabla^2\mathbf{B} = \frac{\mu_0 n_s e^2}{m}\mathbf{B} + \mu_0 \nabla \times \mathbf{c} \tag{8.3.8}$$

but what to do with the integration constant $\mathbf{c}$?

This is the crucial difference between a perfect conductor and a superconductor. For a perfect conductor, we are not be able to assert that the magnetic field is expelled, since $\mathbf{c}$ is not necessarily null. Instead, we can only state that:

$$\nabla^2\left(\frac{\partial \mathbf{B}}{\partial t}\right) = \frac{\mu_0 n_{pc} e^2}{m}\frac{\partial \mathbf{B}}{\partial t} \tag{8.3.9}$$

so the change in magnetic field decays exponentially inside the perfect conductor, as expected by our discussion of persistent magnetic flux.

For superconductors instead, we can explain the Meissner effect only by setting $\mathbf{c} = 0$. Then, we can define the **penetration depth** as:

$$\lambda = \sqrt{\frac{m}{\mu_0 e^2 n_s}} \tag{8.3.10}$$

so that for a superconductor:

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda^2}\mathbf{B} \tag{8.3.11}$$

and thus:

$$\boxed{\nabla \times \mathbf{J}_s = -\frac{e^2 n_s}{m}\mathbf{B}} \tag{8.3.12}$$

which is known as the **second London equation**.

Consider the simple one-dimensional configuration of a superconducting material penetrated by a magnetic field $\mathbf{B}_0 = B_0\mathbf{x}$. Then, by symmetry arguments we see that the magnetic field in the superconductor can only depend on z, so that:

$$\frac{\partial^2 \mathbf{B}}{\partial z^2} = \frac{1}{\lambda^2}\mathbf{B} \implies \mathbf{B} = \mathbf{B}_0 e^{-x/\lambda} \tag{8.3.13}$$

where the positive exponent solution was discarded since it is unphysical.

For a model superconductor, where all free electrons are superconducting, we have that $n_s \sim 10^{29}$ m^{-3}. Consequently, the characteristic length of the decay of the magnetic field within the superconductor is:

$$\lambda \tilde{1} 0 nm \tag{8.3.14}$$

which agrees with the experimentally observed Meissner effect.

Note that if we instead take the curl of the second London equation:

$$\nabla \times (\nabla \times \mathbf{J}_s) = -\frac{n_s e^2}{m}(\nabla \times \mathbf{B}) = -\frac{\mu_0 n_s e^2}{m}\mathbf{J}_s \tag{8.3.15}$$

and since $\nabla \times (\nabla \times \mathbf{J}_s = \nabla(\nabla \cdot \mathbf{J}_s) - \nabla^2 \mathbf{J}_s$:

$$\nabla^2 \mathbf{J}_s = \frac{1}{\lambda^2}\mathbf{J}_s \tag{8.3.16}$$

This is known as the **screening current**. For the simple geometry we considered earlier:

$$\mathbf{J}_s = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \left(\frac{B_{0z}\mathbf{y} - B_{0y}\mathbf{z}}{\mu_0 \lambda}\right)e^{-x/\lambda} \tag{8.3.17}$$

so again we see that the screening current flows within a thin layer characterised by λ.

8.4 Types of superconductors

We have two main types of superconductors, type-I superconductors which always expel magnetic fields below the critical field strength, and type-II superconductors which may be penetrated for sufficiently strong fields.

Consider a long cylindrical type-II superconductor inserted in a magnetic field parallel to its axis. Below a field strength B_{c1}, known as the **lower critical field strength**, the applied field is completely excluded. This is known as the **Meissner phase**.

However, as we increase the field above this value, the material will not immediately transition to the normal state, but will form regions of normal material through which vortices of magnetic fields can pass. This phase is known as the **Abrikosov phase**. As the field strength is increased, the normal regions get closer and closer together, until at some **upper critical field strength** B_{c2} the material transitions completely into the normal state.

Transport phenomena

9

9.1 Conservation laws

In this chapter we will be interested in properties of matter that are conserved locally, and the conservation laws that arise as a result. We do so by studying transport phenomena of time-varying electromagnetic fields, that is, the properties that are "transported" by electromagnetic fields.

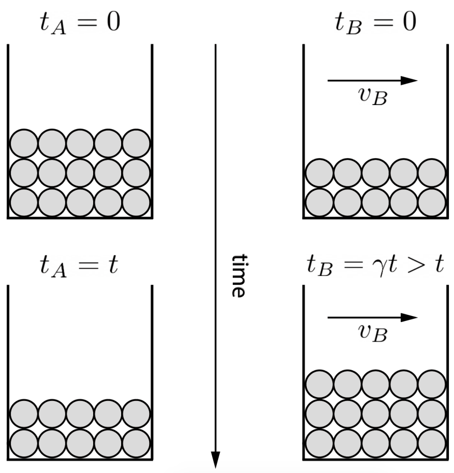


Figure 9.1. Global conservation of "balls" in special relativity

An important point to make before we derive these laws is between global and local conservation. Suppose we have two different reservoirs of balls. Global conservation states that the sum of the number of balls in the two reservoirs remains constant. If the number of balls were to decrease in one at some instant, the number of balls in the other must increase by the same amount in that instant. This however is a flawed view, because from Special relativity we know that simultaneous instants are different for different inertial observers. Hence for charge to be conserved in special relativity we need it to be a local conservation law. We require that for the number of balls in one reservoir to decrease, some amount of balls must have moved out of the reservoir, there must have been a flux of balls.

9.2 Conservation of charge

We revisit the continuity equation which expresses local conservation of charge. If some amount of charge is removed from a region $\mathcal{V}$, then the same amount must have passed in or out through the surface.

Mathematically:

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}', t) d\tau' \tag{9.2.1}$$

Then, the current through region is just the rate of change of the charge enclosed in this region, that is:

$$\frac{dQ}{dt} = \int_{\mathcal{V}} \frac{\partial}{\partial t} \rho(\mathbf{r}', t) d\tau' = -\int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} \tag{9.2.2}$$

$$= -\int_{\mathcal{V}} \nabla \cdot \mathbf{J} \cdot d\tau' \tag{9.2.3}$$

$$\tag{9.2.4}$$

Since this is true for any arbitrary volume $\mathcal{V}$, we find that

$$\boxed{\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0} \tag{9.2.5}$$

which is the famous continuity equation.

It turns out that under appropriate conditions one can write similar local conservation laws for energy and momentum. The general theme will be that if "something" of the EM field decreases in a region, then there must have been a flux of this "something" through the surface boundary of this region.

9.3 Conservation of energy

We have that the energy flux into a given region of space minus the work done on the charges gives the total increase in the electromagnetic fields. Recall that:

$$W_e = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} E^2 d\tau \tag{9.3.1}$$

$$W_m = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} B^2 d\tau \tag{9.3.2}$$

are the energies stored in the electric and magnetic fields. We can define more generally the electromagnetic energy density as:

$$u = \frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2) \tag{9.3.3}$$

so that $ud\tau$ gives the energy stored in the volume $d\tau$. Now the work done on an infinitesimal charge dq as they move in some interval of time dt is:

$$d\mathbf{F} \cdot d\mathbf{l} = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt = dqdt\mathbf{E} \cdot \mathbf{v} \tag{9.3.4}$$

so the work done on all the charges in time dt is:

$$dW = \int dt\mathbf{E} \cdot \mathbf{v}dq \tag{9.3.5}$$

$$= \int dt\mathbf{E} \cdot \mathbf{v}\rho d\tau' \tag{9.3.6}$$

$$= \int dt\mathbf{E} \cdot \mathbf{J}d\tau' \tag{9.3.7}$$

so that:

$$\boxed{\frac{dW}{dt} = \int \mathbf{E} \cdot \mathbf{J}d\tau'} \tag{9.3.8}$$

However, we may also write that:

$$\mathbf{E} \cdot \mathbf{J} = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \varepsilon_0\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t} \tag{9.3.9}$$

and using the vector identity:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{9.3.10}$$

we find that

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\mathbf{B}\frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \tag{9.3.11}$$

$$= -\frac{1}{2}\frac{\partial B^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \tag{9.3.12}$$

Consequently

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) - \frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t} \tag{9.3.13}$$

$$= \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) - \frac{\partial u}{\partial t} \tag{9.3.14}$$

(9.3.8) then becomes:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} ud\tau' - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a} \tag{9.3.15}$$

This equation expresses local conservation of energy, the rate of work done on the charges is equal to the rate of decrease in the fields minus the rate of energy flux in the electromagnetic field through the region.

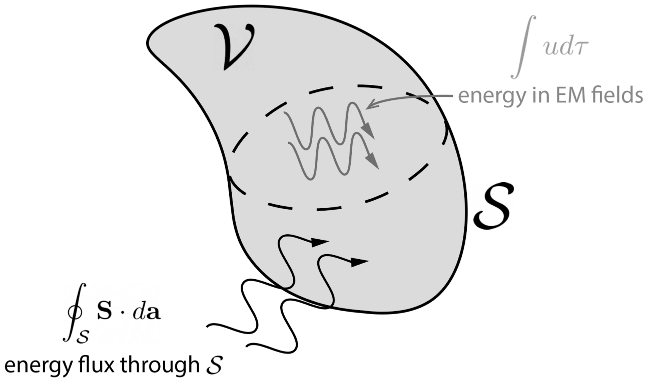


Figure 9.2. Rate of decrease in EM fields minus the flux of energy through a surface gives the power done on sources

So we recognise that:

$$\frac{1}{\mu_0} \oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a} = \text{energy flux through the region} \tag{9.3.16}$$

We define the Poynting vector $\mathbf{S}$ as:

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \implies \text{energy flux through a surface } \mathcal{S} = \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a}} \tag{9.3.17}$$

Hence, (9.5.28) takes its final form:

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} u d\tau' - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a}} \tag{9.3.18}$$

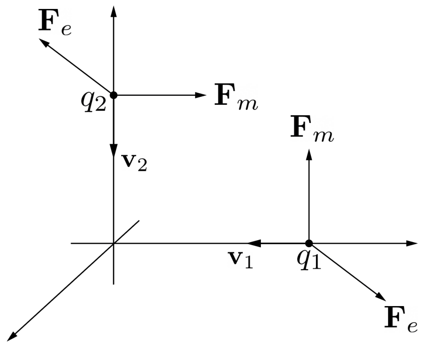
It is important to note that the energy in the fields is not conserved unless $\frac{dW}{dt} = 0$. Of course, in a typical scenario the electromagnetic fields do work on sources that themselves produce electromagnetic fields, and so forth. Energy is transferred from fields to sources and from sources to fields.

If no work is being done on the charges, such as in vacuum, then we get the analogue of the continuity equation:

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0} \tag{9.3.19}$$

9.4 Conservation of momentum

Consider the following paradox. A charge q_1 is placed on the x-axis and moves in the $-x$ direction, another charge q_2 is placed on the y-axis and moves in the $-y$ direction.



The forces on the two charges may be equal in magnitude, but they are not opposite in direction! So does this mean that Newton's third law is violated? No, for the electric and magnetic fields themselves carry momenta with them.

The total force on a localized charge distribution is:

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho d\tau' = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) d\tau' \tag{9.4.1}$$

$$= \int_{\mathcal{V}} \varepsilon_0 \mathbf{E}(\nabla \cdot \mathbf{E}) d\tau + \int_{\mathcal{V}} \left(\frac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B} d\tau' \tag{9.4.2}$$

Now:

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \tag{9.4.3}$$

$$= \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}) \tag{9.4.4}$$

Consequently, the force per unit volume is:

$$\mathbf{f} = \varepsilon_0(\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E})) - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) \tag{9.4.5}$$

Now we have that:

$$[\mathbf{B} \times (\nabla \times \mathbf{B})]_i = \epsilon_{ijk} B_j \epsilon_{klm} \partial_l B_m \tag{9.4.6}$$

$$= \delta_{il}\delta_{jm} B_j \partial_l B_m - \delta_{im}\delta_{jl} B_j \partial_l B_m \tag{9.4.7}$$

$$= B_j \partial_i B_j - B_j \partial_j B_i \tag{9.4.8}$$

$$= \frac{1}{2}\partial_i (B_j B_j) - B_j \partial_j B_i \tag{9.4.9}$$

$$= \left[\frac{1}{2} \nabla B^2 - (\mathbf{B} \cdot \nabla)\mathbf{B} \right]_i \tag{9.4.10}$$

and similarly for $\mathbf{E}$. Hence:

$$\mathbf{f} = \varepsilon_0(\mathbf{E}(\nabla\cdot\mathbf{E}) + (\mathbf{E}\cdot\nabla)\mathbf{E}) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla\left(\varepsilon_0 E^2 - \frac{1}{\mu_0}B^2\right) - \varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) \tag{9.4.11}$$

One may recognize the energy and energy flux densities in the above formula to finally get:

$$\mathbf{f} = \varepsilon_0(\mathbf{E}(\nabla\cdot\mathbf{E}) + (\mathbf{E}\cdot\nabla)\mathbf{E}) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla u - \varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} \tag{9.4.12}$$

We may further simplify (9.4.12) by defining the Maxwell stress tensor:

$$T_{ij} = \varepsilon_0\left(E_iE_j + \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j + \frac{1}{2}\delta_{ij}B^2\right) \tag{9.4.13}$$

and noting that:

$$(\nabla\cdot\mathbf{T})_j = \varepsilon_0\left(E_j\partial_iE_i + E_i\partial_iE_j + \frac{1}{2}\partial_jE^2\right) + \frac{1}{\mu_0}\left(B_j\partial_iB_i + B_i\partial_iB_j + \frac{1}{2}\partial_jB^2\right) \tag{9.4.14}$$

$$= \varepsilon_0\left(E_j\nabla\cdot\mathbf{E} + (\mathbf{E}\cdot\nabla)E_j + \frac{1}{2}\partial_jE^2\right) + \frac{1}{\mu_0}\left(\mathrm{B}\cdot\nabla)B_j + \frac{1}{2}\partial_jB^2\right) \tag{9.4.15}$$

$$\iff \mathbf{f} = \nabla\cdot\mathbf{T} - \varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} \tag{9.4.16}$$

The total force is then:

$$\boxed{\mathbf{F} = \int_{\mathcal{S}}\mathbf{T}\cdot d\mathbf{a} - \varepsilon_0\mu_0\int_{\mathcal{V}}\mathbf{S}d\tau} \tag{9.4.17}$$

The first term is the momentum flow through the region per unit time. The second term instead is the momentum stored in the fields. The physical interpretation of $\mathbf{T}$ is then the force along i per unit area acting on the surface normal to j, known as stress, acting on the system.

Using Newton's second law we find that:

$$\frac{d\mathbf{p}}{dt} = \int_{\mathcal{S}}\mathbf{T}\cdot d\mathbf{a} - \varepsilon_0\mu_0\int_{\mathcal{V}}\mathbf{S}d\tau \tag{9.4.18}$$

where we may recognize he momentum density in the fields as the argument of the second integral in (9.4.17):

$$\mathbf{g} = \varepsilon_0\mu_0\mathbf{S} \tag{9.4.19}$$

For systems with no external forces, we must have that:

$$\int_{\mathcal{S}}\mathbf{T}\cdot d\mathbf{a} - \varepsilon_0\mu_0\int_{\mathcal{V}}\mathbf{S}d\tau \implies \int_{\mathcal{V}}\nabla\cdot\mathbf{T}d\tau - \varepsilon_0\mu_0\int_{\mathcal{V}}\mathbf{S}d\tau \tag{9.4.20}$$

so that we reach conservation of momentum:

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \mathbf{T} \tag{9.4.21}$$

Finally, we may define the angular momentum density as:

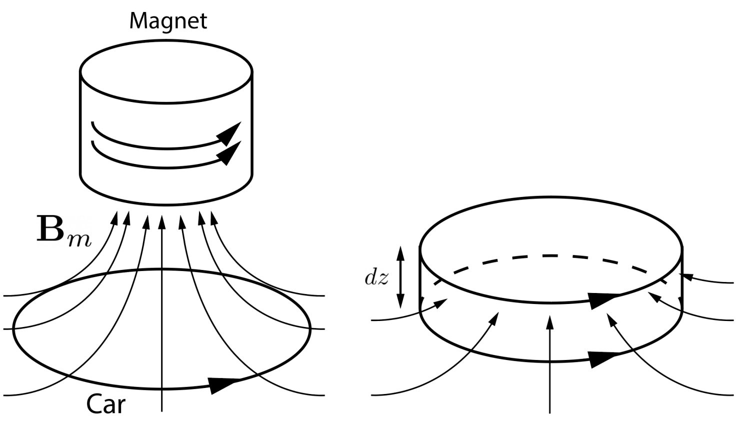
$$\mathbf{l} = \mathbf{r} \times \mathbf{g} = \varepsilon_0 \mu_0 \mathbf{r} \times \mathbf{S} \tag{9.4.22}$$

9.5 Do magnetic fields do work?

We now revisit the claim that magnetic fields do no work, as it is subject to some revision.

Indeed, we know that the Lorentz force law forbids the magnetic field from doing any work on current distributions, including dipoles which are a result of currents. Intrinsic dipoles are a separate case as they do not have a current associated to them, and magnetic fields can indeed perform work on such quantum objects, as was seen in Volume 1. In this section however we will concentrate on classical objects, so configurations that may be related to some current.

Consider a magnetic crane lifting a car. We know that magnetic fields do no work, so how come the car gets lifted up, who does the work?



Well, if we model the car as a circular current loop with charge density λ then the force on the car will be:

$$F = 2\pi I a B_m = 2\pi\lambda\omega a^2 B_m \tag{9.5.1}$$

where B_m is the magnetic field due to the crane. Hence the work dW done in lifting the car by dz is:

$$dW = Fdz = 2\pi\omega\lambda a^2 B_m dz \tag{9.5.2}$$

When doing this, the induced emf in the current loop will be:

$$\varepsilon = -\frac{d\Phi}{dt} = -2\pi a B_m \frac{dz}{dt} \tag{9.5.3}$$

which we can equate to the force per unit charge integrated over the loop:

$$-2\pi a B_m \frac{dz}{dt} = \int \mathbf{f} \cdot d\mathbf{l} = 2\pi a f \implies f = -B_m \frac{dz}{dt} \tag{9.5.4}$$

so over a segment of length dl

$$F = -B_m \frac{dz}{dt} \lambda dl \tag{9.5.5}$$

which will produce a torque slowing the rotation of the loop:

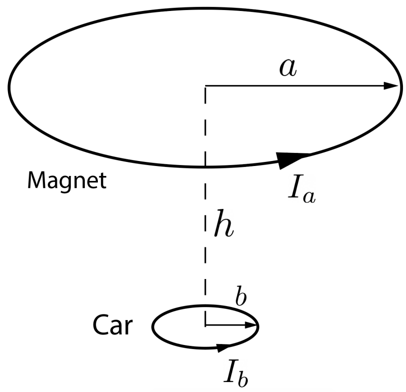
$$\tau = -B_m \frac{dz}{dt} a\lambda \oint dl = -2\pi a^2 \lambda B_m \frac{dz}{dt} \tag{9.5.6}$$

The work done by this torque is:

$$dW_\tau = \tau d\phi = \tau\omega dt = -2\pi a^2 \omega\lambda B_m dz = -dW \tag{9.5.7}$$

which is precisely the opposite of work done in raising the car. Consequently, the reduction in the current through the car (rotational energy) is responsible for the car being lifted up.

Suppose then that we connect the current loop so that the current through it remains constant. We must then have that the work done by the source to maintain the current will be equal to the work done in lifting the current loop.



Let us model the magnet as a circular ring of radius a and current I_a, and the car as a much smaller ring of radius b and current I_b, as well as mass m.

Then the force on the car will be:

$$F_{mag} = \frac{3\pi}{2} \mu_0 I_a I_b \frac{a^2 b^2 h}{(a^2 + h^2)^{5/2}} = mg \tag{9.5.8}$$

so the work done against gravity will be:

$$dW_g = mgdz = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2b^2h}{(a^2+h^2)^{5/2}}dz \tag{9.5.9}$$

The flux through the ring is:

$$\Phi_a = MI_b = \frac{\pi\mu_0}{2}\frac{a^2b^2I_b}{(a^2+h^2)^{3/2}} \tag{9.5.10}$$

so as it rises an induced motional emf will be produced:

$$\begin{aligned}
\varepsilon_a = -\frac{d\Phi_a}{dt} &= -\frac{d\Phi_a}{dh}\frac{dh}{dt} &\quad (9.5.11)\\
&= -\frac{3}{2}\frac{\pi\mu_0}{2}\frac{a^2b^2I_b}{(a^2+h^2)^{5/2}}\cdot 2h\Big(-\frac{dz}{dt}\Big) &\quad (9.5.12)\\
&= -\frac{3\pi\mu_0}{2}\frac{I_b a^2b^2h}{(a^2+h^2)^{5/2}}\frac{dz}{dt} &\quad (9.5.13)
\end{aligned}$$

The work done by the power supply will then be:

$$dW_a = -\varepsilon_a I_a dt = \frac{3\pi\mu_0}{2}\frac{I_a I_b a^2b^2h}{(a^2+h^2)^{5/2}}dz = dW_g \tag{9.5.14}$$

which is equal to the work done against gravity, as required.

It is interesting to note that we also have an induced emf in loop b:

$$\Phi_b = MI_a \implies \varepsilon_b = -I_a\frac{dM}{dt} \tag{9.5.15}$$

and hence to maintain the current I_b constant the source must additionally do work:

$$dW_b = -\varepsilon_b I_b dt = I_a I_b \frac{dM}{dt} = dW_g \tag{9.5.16}$$

So it seems like the power supply actually does $2dW_g$. This is indeed true, the additional dW_g contribution is used to increase the energy stored in the EM fields. Indeed:

$$U = \frac{1}{2}L_a I_a^2 + \frac{1}{2}L_b I_b^2 + MI_aI_b \implies dU = I_aI_b\frac{dM}{dt} = dW_g \tag{9.5.17}$$

as required.

To summarize, magnetic fields can do work on intrinsic magnetic moments, such as an electron's spin. In such cases the Lorentz force law turns into:

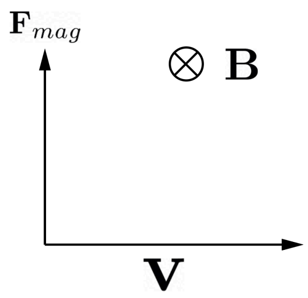
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \nabla(\mathbf{m}\cdot\mathbf{B}) \tag{9.5.18}$$

where **m** is this natural magnetic moment.

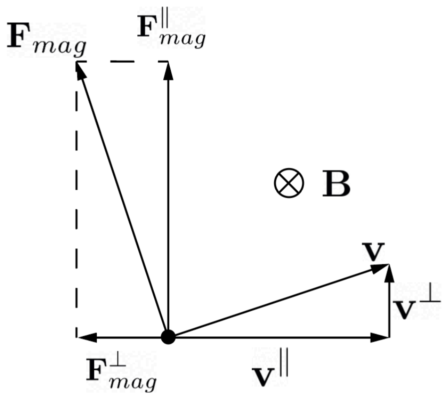
It cannot however do work on any system with an associated current and no intrinsic magnetic moment. Phenomena where magnetic fields seem to be doing work may be explained by examining any external agents that may be instead doing the work (such as batteries) or by considering induced electric fields doing the work.

What if we instead don't have a source maintaining the currents constant?

As the diagram below shows, the initial magnetic force will be upwards, perpendicular to the current. No work is being done. Since the charge carriers are constrained to move in the wire, they will exert a force (not magnetic in nature) on the wire. Hence the wire will start moving upwards.



Now the instant the wire starts moving upwards, the charges will also gain a velocity component $\mathbf{v}^{\perp}$ perpendicular to their original motion. This component will produce a horizontal component $\mathbf{F}^{\perp}_{mag}$ of the magnetic force opposing the flow of current. In the absence of a source then, the currents will decrease, and the work done by the magnetic field in doing so will precisely balance out the work done by the horizontal component in raising the ring. Hence the magnetic field will have done no work, and of course we see this visually because the total force $\mathbf{F}_{mag}$ is indeed perpendicular to the total velocity $\mathbf{v}$.



Let's prove this result. The magnetic fluxes through the two loops are:

$$\Phi_a = L_a I_a + M I_b \tag{9.5.19}$$
$$\Phi_b = L_b I_b + M I_a \tag{9.5.20}$$

which induce emfs in the loops opposing flow of current:

$$\varepsilon_a = -L_a \frac{dI_a}{dt} - \frac{dM}{dt} I_b - M \frac{dI_b}{dt} \tag{9.5.21}$$
$$\varepsilon_b = -L_b \frac{dI_b}{dt} - \frac{dM}{dt} I_a - M \frac{dI_a}{dt} \tag{9.5.22}$$

So in the absence of an external supply of energy we will find that the currents decrease. The rate at which work is done by these emfs is then:

$$\frac{dW_a}{dt} = -L_a \frac{dI_a}{dt} I_a - \frac{dM}{dt} I_b I_a - M \frac{dI_b}{dt} I_a \tag{9.5.23}$$
$$\frac{dW_b}{dt} = -L_b \frac{dI_b}{dt} I_b - \frac{dM}{dt} I_b I_a - M \frac{dI_a}{dt} I_b \tag{9.5.24}$$

Note that the work dW_b is totally due to induced electric fields, the emf comes from Faraday's law. Instead, the work dW_a is partly a faraday emf due to the changing magnetic fields of the two loops (the term $-L_a \frac{dI_a}{dt} I_a - M \frac{dI_b}{dt} I_a$) and partly a motional emf $-\frac{dM}{dt} I_b I_a$ which is done by the horizontal component of the magnetic force. The total work done by the emfs is:

$$\frac{dW_a}{dt} + \frac{dW_b}{dt} = -L_a \frac{dI_a}{dt} I_a - L_b \frac{dI_b}{dt} I_b - 2\frac{dM}{dt} I_a I_b - M I_b \frac{dI_a}{dt} + M I_a \frac{dI_b}{dt} \tag{9.5.25}$$

Meanwhile, the change in the field energy is:

$$U = \frac{1}{2} L_a I_a^2 + \frac{1}{2} L_b I_b^2 + M I_a I_b \tag{9.5.26}$$
$$\frac{dU}{dt} = L_a I_a \frac{dI_a}{dt} + L_a I_b \frac{dI_b}{dt} + M I_b \frac{dI_a}{dt} + M I_a \frac{dI_b}{dt} + \frac{dM}{dt} I_a I_b \tag{9.5.27}$$

Hence we may write that

$$\frac{dU}{dt} + I_a I_b \frac{dM}{dt} = -\left(\frac{dW_a}{dt} + \frac{dW_b}{dt}\right) \tag{9.5.28}$$

We now recognise the work W_g done in lifting the loop in (9.5.28):

$$\frac{dW_g}{dt} = I_a I_b \frac{dM}{dt} \tag{9.5.29}$$

so that:

$$\frac{dU}{dt} + \frac{dW_g}{dt} = -\left(\frac{dW_a}{dt} + \frac{dW_b}{dt}\right) \tag{9.5.30}$$

This tells us that the work done by the emfs in reducing the currents leads to the lifting of the loop and the change in the field energy, as we found earlier. Hence no total work is being done, as would be required by an isolated system without an external source such as a battery. We know however that the magnetic field did no work as was shown in the derivation of (9.5.7).

We may also rewrite (9.5.30) in the energy conservation form:

$$U + W_g + W_a + W_b = \text{cnst.} \tag{9.5.31}$$

So in the absence of a source the work done by the emfs in reducing the currents is paid back by a change in potential energy and kinetic energy (given by W_g using the work-energy theorem).

Note that the work done by the magnetic field's vertical component on the loop is only the $-I_a I_b \frac{dM}{dt}$ which is incidentally the negative of the work done by the horizontal component dW_g. Hence, as required, the magnetic field does no work. The faraday emfs in W_a and W_b instead are responsible for the change in field energy U. What force was responsible for the loop being lifted? The vertical component of the magnetic force, of course, but did the magnetic force do work overall? Of course not.

It may be helpful to compare this situation to an incline plane system. Suppose we a box on an incline plane, and I exert a force on it. The box will then have a net vertical displacement. Is the normal force doing the work? Of course not, because even though it has a vertical component (which is what lifts the box), it has a horizontal component which does opposite work. Overall the normal force therefore has done no work, and we see this because it is perpendicular to the box's overall velocity.

The magnetic field also does no work on loop b, as the emf there is purely faraday. Hence the work is done by an induced electric field rather than a magnetic field.

In the case where a source is present then the conserved quantity of motion is:

$$U + W_g - W_a - W_b = \text{cnst.} \tag{9.5.32}$$

where $W_a + W_b$ is the work done by the source. Again we will have a change in potential and kinetic energy of the small current loop, but this time it won't be accounted for by a decrease in rotational kinetic energy but rather an external source of work, the battery.

Electromagnetic Waves

10

10.1 Waves in 1D

A wave may be defined as a perturbation of some medium which propagates at a fixed shape with constant velocity.

Consequently, suppose we represent the perturbation of the medium with some function $f(z,t)$, and let $f(z,0) = g(z)$. Since the perturbation has a fixed shape, but simply moves with constant velocity, we have that at later times, the displacement at z is really just the initial displacement at $z - vt$, so:

$$f(z,t) = f(z - vt, t) = g(z - vt) \tag{10.1.1}$$

It follows that waves are described by functions of $z - vt$ only.

Consider a stretched string for example. Let us divide the string into infinitesimal segments of length dz approximately, as shown below.

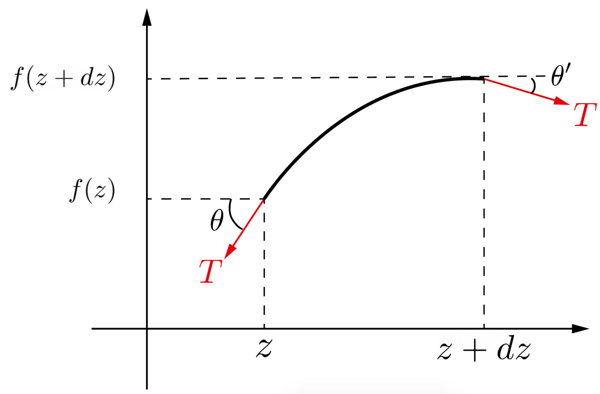


Figure 10.1. Tension in infinitesimal string segment

We then find that

$$\Delta F_y = T(\sin\theta' - \sin\theta) \tag{10.1.2}$$

and using the small angle approximation:

$$\Delta F_y \approx T(\tan\theta' - \tan\theta) \tag{10.1.3}$$
$$= T\left(\frac{\partial f}{\partial z}(z+dz) - \frac{\partial f}{\partial z}(z)\right) \tag{10.1.4}$$
$$= Tdz\frac{\left(\frac{\partial f}{\partial z}(z+dz) - \frac{\partial f}{\partial z}(z)\right)}{dz} \tag{10.1.5}$$
$$= T\frac{\partial^2 f}{\partial z}dz \tag{10.1.6}$$

Using Newton's second law, we may express $\Delta F_y = \mu dz \frac{\partial^2 f}{\partial t}$ where μ is the linear density of the rope. Inserting this into (10.1.6) we finally get:

$$\frac{\partial^2 f}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 f}{\partial z^2} \tag{10.1.7}$$

Define $v = \sqrt{\frac{T}{\mu}}$ as the characteristic velocity of this problem. Then:

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}} \tag{10.1.8}$$

This is known as the wave-equation. Indeed, it admits solutions $f(z,t) = g(z-vt) \equiv g(z')$ as solutions:

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 g}{\partial z'^2} \tag{10.1.9}$$
$$\frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 g}{\partial z'^2} = v^2\frac{\partial^2 f}{\partial z^2} \tag{10.1.10}$$

as desired.

Now since the wave-equation does not depend on v but rather v^2, the sign of the velocity (direction) does not matter. Hence the wave-equation also admits solutions of the form $f(z,t) = h(z+vt)$, which represent waves travelling in the $-z$ direction. The most general solution to the wave-equation is then a linear superposition of the two:

$$F(z,t) = Ag(z-vt) + Bh(z+vt) \tag{10.1.11}$$

Now the most famous solution to the wave-equation is the sinusoidal solution

$$f(z,t) = A\cos(k(z-vt)+\phi) = A(\cos kz - \omega t + \phi),\ \omega = kv \tag{10.1.12}$$

or alternatively:

$$f(z,t) = A\cos(k(z+vt)+\phi) = A(\cos kz - \omega t + \phi),\ \omega = kv \tag{10.1.13}$$

For reasons that will become apparent when discussing boundary conditions, it is easier if we use the parity of the cosine function to write:

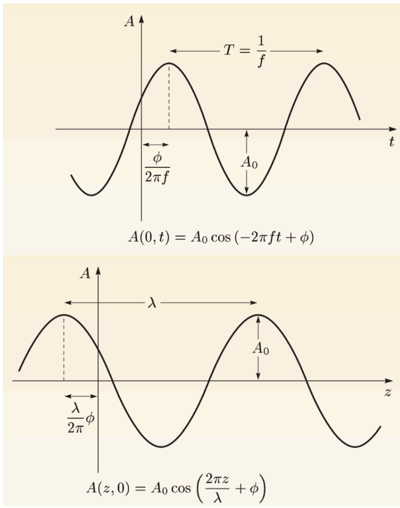
$$f(z,t) = A\cos(-k(z+vt)+\phi) = A(\cos kz - \omega t + \phi),\ \omega = kv \tag{10.1.14}$$

We can write an associated complex wave function:

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)},\ \tilde{A} = Ae^{i\phi} \tag{10.1.15}$$

which is related to f by:

$$f(z,t) = \mathrm{Re}\,\tilde{f}(z,t) \tag{10.1.16}$$



We define:

(i) Amplitude: A

(ii) Angular frequency: ω

(iii) Wavenumber: k

(iv) Phase shift: ϕ

(v) Frequency: $f = \frac{\omega}{2\pi}$

(vi) Wavelength: $\lambda = \frac{2\pi}{k}$

10.2 1D-boundary conditions

Now suppose we attach two massless strings at $z = 0$, with linear densities μ_1 ad μ_2, and hence wave velocities v_1 and v_2 respectively. Suppose an initial perturbation forms an incident wave:

$$\widetilde{f}_I(z,t) = \widetilde{A}_I e^{ik_1 z - \omega t} \tag{10.2.1}$$

When this wave meets the boundary between the two strings, it will split into a reflected wave

$$\widetilde{f}_R(z,t) = \widetilde{A}_R e^{ik_2 z + \omega t} \tag{10.2.2}$$

and a transmitted wave

$$\widetilde{f}_T(z,t) = \widetilde{A}_T e^{ik_2 z - \omega t} \tag{10.2.3}$$

but what are the appropriate boundary conditions?

Firstly, since the two strings are tied together, there can be no discontinuity at $z = 0$:

$$\widetilde{f}_I(0,t) + \widetilde{f}_R(0,t) = \widetilde{f}_T(0,t) \tag{10.2.4}$$

Also, we cannot have discontinuities in $\frac{\partial \widetilde{f}}{\partial z}(0,t)$. Indeed, if there were a discontinuity then the resulting force ΔF_y we derived in (10.1.3) would cause the massless string to accelerate infinitely. Hence:

$$\frac{\partial \widetilde{f}_I}{\partial z}(0,t) + \frac{\partial \widetilde{f}_R}{\partial z}(0,t) = \frac{\partial \widetilde{f}_T}{\partial z}(0,t) \tag{10.2.5}$$

Here we see why we wrote $\cos(-kz-\omega t)$ instead of $\cos(kz+\omega t)$ when discussing waves moving to the right/left. The latter choice would create some problems since one partial derivative would have a $\sin \omega t$ term whereas the other would contain a $\sin -\omega t$. It is simpler if we just let the time component have the same signs.

Applying these boundary conditions we find that:

$$\widetilde{A}_I + \widetilde{A}_R = \widetilde{A}_T \tag{10.2.6}$$

and

$$k_1(\widetilde{A}_I - \widetilde{A}_R) = k_2 \widetilde{A}_T \tag{10.2.7}$$

These can be solved to give:

$$\widetilde{A}_T = \frac{2k_1}{k_1+k_2}\widetilde{A}_I \tag{10.2.8}$$

$$\widetilde{A}_R = \frac{k_1-k_2}{k_1+k_2}\widetilde{A}_I \tag{10.2.9}$$

and thus:

$$A_T = \frac{2k_1}{k_1+k_2}A_I e^{i(\delta_I - \delta T} \tag{10.2.10}$$

$$A_R = \frac{k_1-k_2}{k_1+k_2}A_I e^{i(\delta_I - \delta R} \tag{10.2.11}$$

Since A_R, A_I and A_T must be positive real numbers, only certain combinations of phases will be physically allowed. More specifically:

(i) if $\mu_1 > \mu_2$ then $k_1 > k_2$ and hence $\delta_I = \delta_R = \delta_T$

(ii) if $\mu_2 > \mu_1$ then $k_1 < k_2$ and hence $\delta_R + \pi = \delta_I = \delta_T$, so reflected wave will be upside down.

Interestingly, in the limit as $\mu_2 \to \infty$, we have that $A_T \to 0$ so we only get a reflected wave, which is actually upside-down.

10.3 Electromagnetic Waves in vacuum

We present Maxwell's equations in a vacuum:

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E} &= 0 && (10.3.1a) \\ \nabla \cdot \mathbf{B} &= 0 && (10.3.1b) \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (10.3.1c) \\ \nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} && (10.3.1d) \end{aligned}}$$

Taking the curl of the third equation:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{10.3.2}$$

$$\iff \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla^2 \mathbf{E} \tag{10.3.3}$$

$$\iff \nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{10.3.4}$$

Of we let $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ then we get the wave-equation:

$$\boxed{\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}} \tag{10.3.5}$$

One similarly gets:

$$\boxed{\nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}} \tag{10.3.6}$$

So oscillations in electric and magnetic fields will produce an electromagnetic wave. What is perhaps even more fascinating is that the approximate value of c is $3.00 \times 10^8 m/s$, which matches the speed of light! So light must be an electromagnetic wave.

Note that we are dealing with electromagnetic fields in vacuum, that is, in empty space with no charge and current sources. Light can therefore exist in vacuum, unlike sound (which requires a medium of propagation). This means that the electromagnetic wave does not have a medium of propagation, it is simply an oscillation in the electric and magnetic fields.

Let us consider monochromatic plane wave solutions. These are waves of a single frequency (monochromatic) travelling in a single direction, which we label z, and uniform over the plane perpendicular to the direction of propagation (plane). That is, it has no $x - y$ dependence.

Then we may set:

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \implies \mathbf{E} = \mathbf{E}_0 \cos(kz - \omega t) \tag{10.3.7}$$

which represents a plane wave travelling in the z direction, with polarization along $\widetilde{\mathbf{E}_0}$. To check that this is indeed a solution to the wave-equation, let us assume that $\mathbf{E}_0 = E_0\hat{\mathbf{x}}$, we find that:

$$-E_0 k^2 \cos(kz - \omega t) = -\frac{\omega^2}{c^2}\cos(kz - \omega t) \implies \boxed{\omega = ck} \tag{10.3.8}$$

Therefore, we require $\omega = ck$ for this to be an electromagnetic wave. But what happened to the magnetic field? We can use Faraday's law to recover an expression for $\widetilde{\mathbf{B}}$:

$$\nabla \times \mathbf{E} = -kE_0 \sin(k_z - \omega t)\hat{\mathbf{y}} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.3.9}$$

and integrating over time:

$$\mathbf{B} = \frac{k}{\omega}E_0 \cos(kz - \omega t)\hat{\mathbf{y}} = \frac{1}{c}E_0\cos(kz - \omega t)\hat{\mathbf{y}} \tag{10.3.10}$$

This is also a plane wave, just with different magnitude (much much larger magnitude for the electric field). Also, we notice that the direction of propagation of the electromagnetic wave is perpendicular to the direction of the perturbations in $\mathbf{E}$ and $\mathbf{B}$. Hence we may conclude that electromagnetic waves are transverse waves.

More generally, for an electromagnetic plane wave travelling in an arbitrary direction along the wave vector $\hat{\mathbf{k}}$ with polarization $\hat{\mathbf{n}}$:

$$\widetilde{\mathbf{E}} = \widetilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}} \tag{10.3.11}$$

with dispersion relation $\omega = c|\mathbf{k}|$.

Again let us try to find the associated magnetic field. Using Faraday's law we find that:

$$ie^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\widetilde{E}_{0z}k_y - \widetilde{E}_{0y}k_z) = -\frac{\partial \widetilde{B}_x}{\partial t} \tag{10.3.12}$$

$$-ie^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\widetilde{E}_{0z}k_x - \widetilde{E}_{0x}k_z) = -\frac{\partial \widetilde{B}_y}{\partial t} \tag{10.3.13}$$

$$ie^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\widetilde{E}_{0y}k_x - \widetilde{E}_{0x}k_y) = -\frac{\partial \widetilde{B}_x}{\partial t} \tag{10.3.14}$$

so that:

$$ie^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\mathbf{k} \times \widetilde{\mathbf{E}}_0) = -\frac{\partial \widetilde{\mathbf{B}}}{\partial t} \implies \boxed{\widetilde{\mathbf{B}} = \frac{1}{\omega}\mathbf{k} \times \mathbf{E} = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}} \tag{10.3.15}$$

and similarly:

$$\boxed{\widetilde{\mathbf{E}} = c\widetilde{\mathbf{B}} \times \hat{\mathbf{k}}} \tag{10.3.16}$$

We also have that imposing Gauss' law and the no-monopole law:

$$\nabla \cdot \widetilde{\mathbf{E}} = (E_{0x}k_x + E_{0y}k_y + E_{0z}k_z)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = 0 \implies \widetilde{\mathbf{E}}_0 \cdot \mathbf{k} = 0 \tag{10.3.17}$$

$$\nabla \cdot \widetilde{\mathbf{B}} = (B_{0x}k_x + B_{0y}k_y + B_{0z}k_z)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = 0 \implies \widetilde{\mathbf{B}}_0 \cdot \mathbf{k} = 0 \tag{10.3.18}$$

so we do indeed find that the direction of propagation $\mathbf{k}$ is perpendicular to to direction of perturbation $\widetilde{\mathbf{E}}_0$ and $\widetilde{\mathbf{B}}_0$, that is, electromagnetic waves are transversal.

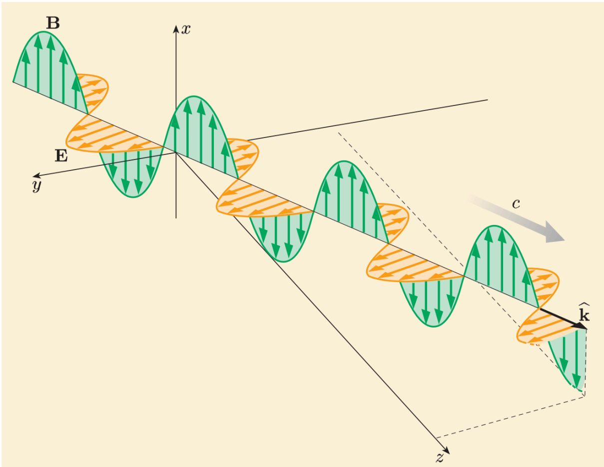


Figure 10.2. Caption

10.4 Energy transported by EM waves

10.5 Reflection of waves in vacua

Consider an electromagnetic wave incident on a perfect conductor set at $z = 0$:

$$\widetilde{\mathbf{E}}_I = \frac{\widetilde{E}_0}{2}e^{i(kz-\omega t)}\hat{\mathbf{x}} \tag{10.5.1}$$

with corresponding magnetic field

$$\widetilde{\mathbf{B}}_I = \frac{\widetilde{E}_0}{2c}e^{i(kz-\omega t)}\hat{\mathbf{y}} \tag{10.5.2}$$

Since there can be no transmitted wave inside the conductor, we will only get a reflected wave:

$$\widetilde{\mathbf{E}}_R = \frac{\widetilde{E}_{R0}}{2}e^{-i(kz+\omega t)}\hat{\mathbf{x}} \tag{10.5.3}$$

with associated magnetic field

$$\widetilde{\mathbf{B}}_R = -\frac{\widetilde{E}_{R0}}{2c}e^{-i(kz+\omega t)}\hat{\mathbf{y}} \tag{10.5.4}$$

Now from the boundary conditions of electric fields at conducting interfaces we see that:

$$\mathbf{E}(z=0)=\mathbf{0} \tag{10.5.5}$$

Hence:

$$E_{R0}=-E_0 \implies \widetilde{\mathbf{E}}_R=-\frac{\widetilde{E}_0}{2}e^{-i(kz+\omega t)}\hat{\mathbf{x}} \tag{10.5.6}$$

Taking the real part we find:

$$\mathbf{E}_I=\frac{E_0}{2}\cos(kz-\omega t)\hat{\mathbf{x}} \tag{10.5.7}$$

$$\mathbf{E}_R=-\frac{E_0}{2}\cos(-kz-\omega t)\hat{\mathbf{x}} \tag{10.5.8}$$

Now the total electric field is then:

$$\widetilde{\mathbf{E}}=\frac{\widetilde{E}_0}{2}(e^{i(kz-\omega t)}-e^{-i(kz+\omega t)})\hat{\mathbf{x}} \tag{10.5.9}$$

$$=\frac{\widetilde{E}_0}{2}2i\sin(kz)e^{-i\omega t}\hat{\mathbf{x}} \tag{10.5.10}$$

$$=i\widetilde{E}_0\sin(kz)e^{-i\omega t}\hat{\mathbf{x}} \tag{10.5.11}$$

so taking the real part:

$$\boxed{\mathbf{E}=E_0\sin kz\sin\omega t\hat{\mathbf{x}}} \tag{10.5.12}$$

The corresponding magnetic field is similarly:

$$\widetilde{\mathbf{B}}=\frac{\widetilde{E}_0}{2c}(e^{i(kz-\omega t)}+e^{-i(kz+\omega t)})\hat{\mathbf{y}} \tag{10.5.13}$$

$$=\frac{\widetilde{E}_0}{2c}2\cos(kz)e^{-i\omega t}\hat{\mathbf{y}} \tag{10.5.14}$$

$$=\frac{\widetilde{E}_0}{c}\cos(kz)e^{-i\omega t}\hat{\mathbf{y}} \tag{10.5.15}$$

so taking the real part:

$$\boxed{\mathbf{B}=\frac{E_0}{c}\cos kz\cos\omega t\hat{\mathbf{y}}} \tag{10.5.16}$$

These are standing waves!

It is important to note however that these electric and magnetic fields do not satisfy the relation:

$$\mathbf{B}=\frac{1}{\omega}\mathbf{k}\times\mathbf{E}=\frac{1}{c}\hat{\mathbf{k}}\times\mathbf{E} \tag{10.5.17}$$

they are indeed out of phase. This is because (10.5.17) only holds for travelling waves (remember we derived it by considering plane waves).

Let us look at the energy transport of this travelling wave. The Poynting vector, that is, the rate of energy transfer per unit area is:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{E_0^2}{\mu_0 c}\sin\omega t\cos\omega t\cos kz\sin kz\hat{\mathbf{z}} \tag{10.5.18}$$

$$= \frac{E_0^2}{4\mu_0 c}\sin 2\omega t\sin 2kz\hat{\mathbf{z}} \tag{10.5.19}$$

When averaged over time we of course find that:

$$\langle\mathbf{S}\rangle = 0 \tag{10.5.20}$$

as we have a superposition of two waves transporting energy in two opposite directions.

This phenomenon is often used in microwaves, where electromagnetic waves are reflected at a conducting boundary, producing a standing wave. These standing waves oscillate up and down, and therefore vibrating the water molecules inside food which have a tiny dipole moment.

10.6 Circular polarization

Suppose we superpose two plane waves, one polarized along $\hat{\mathbf{x}}$:

$$\widetilde{\mathbf{E}}_x = E_{0x}e^{i(kz-\omega t+\phi_x)}\hat{\mathbf{x}} \tag{10.6.1}$$

and another along $\hat{\mathbf{y}}$:

$$\widetilde{\mathbf{E}}_y = E_{0y}e^{i(kz-\omega t+\phi_y)}\hat{\mathbf{y}} \tag{10.6.2}$$

Suppose that $\phi_x = \phi_y = \phi$, then the superposition of these two waves will give another plane wave:

$$\widetilde{\mathbf{E}} = (E_{0x}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}})e^{i(kz-\omega t+\phi)} \tag{10.6.3}$$

with polarization along $E_{0x}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}}$. Now suppose that $\phi_x = \phi_y + \frac{\pi}{2} = \phi + \frac{\pi}{2}$, then:

$$\widetilde{\mathbf{E}} = E_{0x}\hat{\mathbf{x}}e^{i(kz-\omega t+\phi+\frac{\pi}{2})} + E_{0y}\hat{\mathbf{y}})e^{i(kz-\omega t+\phi)} \tag{10.6.4}$$

$$= (E_{0x}e^{i\frac{\pi}{2}}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}})e^{i(kz-\omega t+\phi)} \tag{10.6.5}$$

$$= (iE_{0x}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}})e^{i(kz-\omega t+\phi)} \tag{10.6.6}$$

Taking the real part:

$$\mathbf{E} = -E_{0x}\sin(kz-\omega t+\phi)\hat{\mathbf{x}} + E_{0y}\cos(kz-\omega t+\phi)\hat{\mathbf{y}} \tag{10.6.7}$$

So for a fixed $z = 0$:

$$\mathbf{E}_0 = -E_{0x}\sin(-\omega t+\phi)\hat{\mathbf{x}} + E_{0y}\cos(-\omega t+\phi)\hat{\mathbf{y}} \tag{10.6.8}$$

the polarization changes over time, it rotates at an angular frequency ω.

This is known as elliptical polarization. If we set $E_{0x} = E_{0y} = E_0$ then we get a special case of elliptical polarization known as circular polarization.

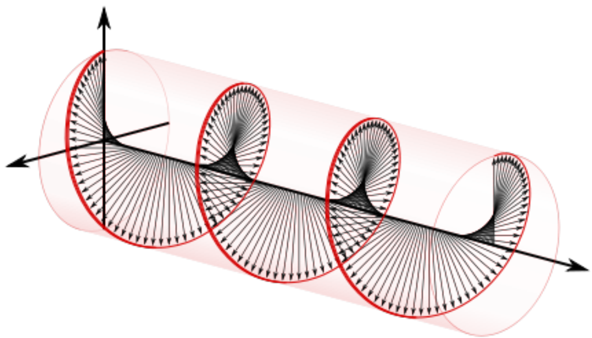


Figure 10.3. Circularly polarized wave

10.7 Electromagnetic waves in LIH media

For general materials Maxwell's equations in the absence of free sources read:

$$\nabla \cdot \mathbf{D} = 0 \tag{10.7.1a}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{10.7.1b}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.7.1c}$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{10.7.1d}$$

We now consider the more special case of an electromagnetic wave travelling through an LIH medium. Maxwell's equations in such materials with permittivity ε and permeability μ read:

$$\nabla \cdot \mathbf{E} = 0 \tag{10.7.2a}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{10.7.2b}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.7.2c}$$
$$\nabla \times \mathbf{B} = \mu\varepsilon\frac{\partial \mathbf{E}}{\partial t} \tag{10.7.2d}$$

in the absence of free sources. Using Hence we will find the very same wave equation:

$$\boxed{\nabla^2 \mathbf{E} = \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \ \nabla^2 \mathbf{B} = \frac{1}{v^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}, \ v = \frac{1}{\sqrt{\mu\epsilon}}} \tag{10.7.3}$$

So waves in media will travel at speed $v = \frac{1}{\sqrt{\mu\epsilon}}$. Defining n, the index of refraction, as:

$$\boxed{n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\mu_r\varepsilon_r}} \tag{10.7.4}$$

where $\mu_r = \frac{\mu}{\mu_0}$ and $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$ are the relative permittivity and permeability respectively. Hence, we find that:

$$\boxed{\omega = vk = \frac{ck}{n}} \tag{10.7.5}$$

10.8 Snell's laws and Fresnel equations

Consider an electromagnetic wave incident on the interface at $z = 0$ between two LIH media, as shown in 10.5 and 10.4. We may model the incoming wave as a plane wave:

$$\widetilde{\mathbf{E}}_I = \widetilde{\mathbf{E}}_{I0}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)},\ \widetilde{\mathbf{B}}_I = \frac{1}{v_1}\hat{\mathbf{k}}_I \times \widetilde{\mathbf{E}}_{I0}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} \tag{10.8.1}$$

which results in a reflected wave:

$$\widetilde{\mathbf{E}}_T = \widetilde{\mathbf{E}}_{R0}e^{i(\mathbf{k}_r\cdot\mathbf{r}+\omega t)},\ \widetilde{\mathbf{B}}_R = \frac{1}{v_1}\hat{\mathbf{k}}_R \times \widetilde{\mathbf{E}}_{R0}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} \tag{10.8.2}$$

and a transmitted wave:

$$\widetilde{\mathbf{E}}_T = \widetilde{\mathbf{E}}_{T0}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)},\ \widetilde{\mathbf{B}}_T = \frac{1}{v_2}\hat{\mathbf{k}}_T \times \widetilde{\mathbf{E}}_{T0}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)} \tag{10.8.3}$$

These must satisfy the boundary conditions (in the absence of free sources σ_f and $\mathbf{J}_f$):

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0 \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \tag{10.8.4}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = 0 \tag{10.8.5}$$

Substituting (10.8.1) into the above we will get equations of the form:

$$Ae^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + Be^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} + Ce^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)} = 0 \tag{10.8.6}$$

where A, B, C are constants we shall determine in a moment. Now it turns out that the only non-trivial solutions to $Ae^{iax} + Be^{ibx} + Ce^{icx} = 0$ occur when $a = b = c$ due to the orthogonality of e^{ikx} functions over $[0, 2\pi]$. Hence we must have that:

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r},\ z = 0 \tag{10.8.7}$$

We deduce that $(k_I)_x = (k_R)_x = (k_T)_x$ as well as $(k_I)_y = (k_R)_y = (k_T)_y$, implying that the three wave-vectors must lie in the same plane as the normal vector, which we have been calling the plane of incidence. This also implies that:

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \implies \boxed{n_I \sin\theta_I = n_R \sin\theta_R = n_T \sin\theta_T} \tag{10.8.8}$$

Now since n_I only depends on μ_r and ε_r, we have that $n_I \sin\theta_I = n_R \sin\theta_R = n_I \sin\theta_R$ implying that $\boxed{\theta_I = \theta_R}$. This is **Snell's first law of reflection**, it states that the incident and reflected waves' wave-vectors form the same angle with the normal to the plane of incidence. Moreover, we also get that $\boxed{n_I \sin\theta_I = n_T \sin\theta_T}$, which is **Snell's second law of refraction**.

Computation of the A, B, C coefficients is a bit more complicated, but it will allow us to derive more useful equations, known as the Fresnel equations. To do so however we must first specify the polarization of the incident wave, which may be perpendicular to the plane of incidence or parallel to it.

Parallel polarization

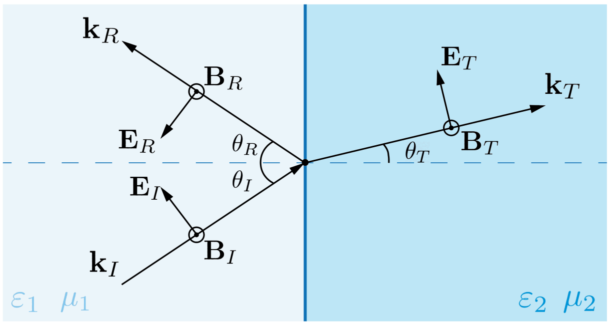


Figure 10.4. Reflection and transmission at LIH interface with parallel polarization.

We firstly find using the boundary conditions for $\mathbf{E}$ that:

$$\varepsilon_1(\widetilde{E}_{I0} + \widetilde{E}_{R0}) \sin\theta_I = \varepsilon_2 \widetilde{E}_{T0} \sin\theta_T \tag{10.8.9}$$

$$\implies \widetilde{E}_{I0} + \widetilde{E}_{R0} = \frac{\varepsilon_2}{\varepsilon_1}\frac{n_I}{n_T}\widetilde{E}_{T0} = \sqrt{\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}} \tag{10.8.10}$$

and

$$(\widetilde{E}_{I0} - \widetilde{E}_{R0}) \cos\theta_I = \widetilde{E}_{T0} \cos\theta_T \tag{10.8.11}$$

Instead, for the magnetic field $\mathbf{B}$ the boundary conditions imply:

$$\frac{1}{\mu_1}(B_{I0} + B_{R0}) = \frac{1}{\mu_2} B_{T0} \tag{10.8.12}$$

$$\iff \frac{1}{v_1\mu_1}(E_{I0} + E_{R0}) = \frac{1}{v_2\mu_2} E_{T0} \tag{10.8.13}$$

Letting $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ and $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \sqrt{\frac{\mu_1\epsilon_2}{\epsilon_1\mu_2}}$ we get that:

$$\widetilde{E}_{I0} - \widetilde{E}_{R0} = \alpha \widetilde{E}_{T0} \tag{10.8.14}$$

$$\widetilde{E}_{I0} + \widetilde{E}_{R0} = \beta \widetilde{E}_{T0} \tag{10.8.15}$$

These have solutions:

$$\widetilde{E}_{R0} = \frac{\beta - \alpha}{\beta + \alpha}\widetilde{E}_{I0}, \quad \widetilde{E}_{T0} = \frac{2}{\beta + \alpha}\widetilde{E}_{I0} \tag{10.8.16}$$

Let us evaluate the Poynting vector for these waves. We have that:

$$\mathbf{S}_I = \frac{1}{\mu_1}\mathbf{E}_I \times \mathbf{B} = \frac{1}{\mu_1 v_1}|E_{I0}|^2 \cos^2(\mathbf{k}_I \cdot \mathbf{r} - \omega t) \tag{10.8.17}$$

and averaging over a period $\frac{2\pi}{\omega}$ we get that:

$$\langle \mathbf{S}_I \rangle = \frac{1}{2\mu_1 v_1}|E_{I0}|^2 \hat{\mathbf{k}}_I \tag{10.8.18}$$

and similarly:

$$\langle \mathbf{S}_R \rangle = \frac{1}{2\mu_1 v_1}|E_{R0}|^2 \hat{\mathbf{k}}_R, \langle \mathbf{S}_t \rangle = \frac{1}{2\mu_2 v_2}|E_{T0}|^2 \hat{\mathbf{k}}_T \tag{10.8.19}$$

Consequently, the average power per unit area (intensity) incident, reflected and trasmitted on the interface are:

$$I_I = \frac{1}{2\mu_1 v_1}|E_{I0}|^2 \hat{\mathbf{k}}_I \cos\theta_I \tag{10.8.20}$$

$$I_R = \frac{1}{2\mu_1 v_1}|E_{R0}|^2 \hat{\mathbf{k}}_I \cos\theta_I \tag{10.8.21}$$

$$I_T = \frac{1}{2\mu_2 v_2}|E_{T0}|^2 \hat{\mathbf{k}}_T \cos\theta_T \tag{10.8.22}$$

We define the transmittance as the ratio of the transmitted and incident intensities, and similarly the reflectance as the ratio of the reflected and incident intensities. Consequently we get that:

$$\boxed{\begin{aligned} R &= \frac{|E_{R0}|^2}{|E_{I0}|^2} = \left(\frac{\beta - \alpha}{\beta + \alpha}\right)^2 && (10.8.23\text{a}) \\ T &= \alpha\beta\frac{|E_{T0}|^2}{|E_{I0}|^2} = \alpha\beta\left(\frac{2}{\beta + \alpha}\right)^2 && (10.8.23\text{b}) \end{aligned}}$$

We firstly note that the incident and reflected waves are in phase if $\beta > \alpha$. Since in most applications $\mu_1 \approx \mu_2$ we have that whenever $n_1 \cos\theta_I > n_2 \cos\theta_T$ the incident and reflected waves will be in phase, whereas when $n_1 \cos\theta_I < n_2 \cos\theta_T$ they will be out of phase by π. The incident and reflected waves are always in phase.

Between these two regimes, we have the case where $\alpha = \beta$ which occurs at a special angle θ_B, known as the **Brewster angle**:

$$\alpha = \frac{\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_I}}{\cos\theta_I} = \beta \tag{10.8.24}$$

$$\iff \sin^2\theta_B = \frac{1 - \beta^2}{\frac{n_1^2}{n_2^2} - \beta^2} \tag{10.8.25}$$

Since in most cases $\mu_1 \approx m_2$ we may take $\beta = \frac{n_2}{n_1}$ and then:

$$\sin^2\theta_B = \frac{\beta^2 - \beta^4}{1-\beta^4} = \frac{\beta^2}{1+\beta^2} \implies \tan\theta_B \approx \beta = \frac{n_2}{n_1} \tag{10.8.26}$$

Perpendicular polarization

Consider the following scenario: We firstly find using the boundary conditions for **E**

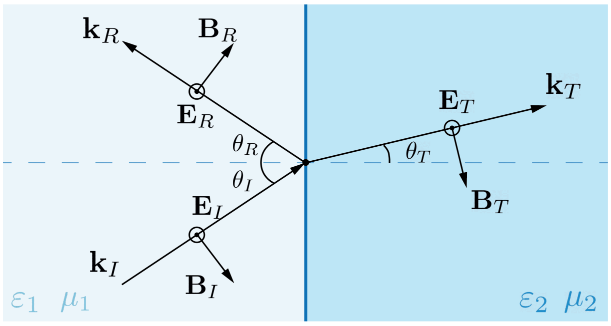


Figure 10.5. Reflection and transmission at LIH interface with perpendicular polarization.

that:

$$(\widetilde{E}_{I0} + \widetilde{E}_{R0}) = \widetilde{E}_{T0} \tag{10.8.27}$$

$$\iff \widetilde{B}_{I0} + \widetilde{B}_{R0} = \frac{v_2}{v_1}\widetilde{B}_{T0} \tag{10.8.28}$$

Instead, for the magnetic field **B** the boundary conditions imply:

$$\frac{1}{\mu_1}(\widetilde{B}_{I0} - \widetilde{B}_{R0})\cos\theta_I = \frac{1}{\mu_2}B_{T0}\cos\theta_T \tag{10.8.29}$$

$$\iff (\widetilde{B}_{I0} - \widetilde{B}_{R0}) = \frac{\mu_1}{\mu_2}\frac{\cos\theta_T}{\cos\theta_I}B_{T0} \tag{10.8.30}$$

and similarly:

$$(\widetilde{B}_{I0} + \widetilde{B}_{R0})\sin\theta_I = \widetilde{B}_{T0}\sin\theta_T \tag{10.8.31}$$

$$\iff \widetilde{B}_{I0} + \widetilde{B}_{R0} = \frac{n_1}{n_2}\widetilde{B}_{T0} \tag{10.8.32}$$

which is equivalent to (10.8.28) since $n = \frac{c}{v}$.

Letting $\gamma = \frac{\mu_1 \cos\theta_T}{\mu_2 \cos\theta_I}$ and $\eta = \frac{v_2}{v_1}$ we get that:

$$\widetilde{B}_{I0} - \widetilde{B}_{R0} = \gamma\widetilde{B}_{T0} \tag{10.8.33}$$

$$\widetilde{B}_{I0} + \widetilde{B}_{R0} = \eta\widetilde{B}_{T0} \tag{10.8.34}$$

These have solutions:

$$\widetilde{B}_{R0} = \frac{\eta - \gamma}{\eta + \gamma}\widetilde{B}_{I0}, \quad \widetilde{B}_{T0} = \frac{2}{\eta + \gamma}\widetilde{B}_{I0} \tag{10.8.35}$$

Let us evaluate the Poynting vector for these waves. We have that:

$$\mathbf{S}_I = \frac{1}{\mu_1}\mathbf{E}_I \times \mathbf{B}_I = \frac{v_1}{\mu_1}|B_{I0}|^2 \cos^2(\mathbf{k}_I \cdot \mathbf{r} - \omega t) \tag{10.8.36}$$

and averaging over a period $\frac{2\pi}{\omega}$ we get that:

$$\langle \mathbf{S}_I \rangle = \frac{v_1}{2\mu_1}|B_{I0}|^2 \hat{\mathbf{k}}_I \tag{10.8.37}$$

and similarly:

$$\langle \mathbf{S}_R \rangle = \frac{v_1}{2\mu_1}|B_{R0}|^2 \hat{\mathbf{k}}_R, \langle \mathbf{S}_T \rangle = \frac{v_2}{2\mu_2}|B_{T0}|^2 \hat{\mathbf{k}}_T \tag{10.8.38}$$

Consequently, the average power per unit area (intensity) incident, reflected and trasmitted on the interface are:

$$I_I = \frac{v_1}{2\mu_1}|B_{I0}|^2 \hat{\mathbf{k}}_I \cos\theta_I \tag{10.8.39}$$

$$I_R = \frac{v_1}{2\mu_1}|B_{R0}|^2 \hat{\mathbf{k}}_I \cos\theta_I \tag{10.8.40}$$

$$I_T = \frac{v_2}{2\mu_2}|B_{T0}|^2 \hat{\mathbf{k}}_T \cos\theta_T \tag{10.8.41}$$

We define the transmittance as the ratio of the transmitted and incident intensities, and similarly the reflectance as the ratio of the reflected and incident intensities. Consequently we get that:

$$\boxed{R = \frac{|B_{0R}|^2}{|B_{I0}|^2} = \left(\frac{\eta - \gamma}{\eta + \gamma}\right)^2} \tag{10.8.42a}$$

$$\boxed{T = \gamma\eta\frac{|B_{T0}|^2}{|B_{I0}|^2} = \gamma\eta\left(\frac{2}{\eta + \gamma}\right)^2} \tag{10.8.42b}$$

We firstly note that the incident and reflected waves are in phase if $\eta > \gamma$. Since in most applications $\mu_1 \approx \mu_2$ we have that whenever $n_2 \cos\theta_I > n_1 \cos\theta_T$ the incident and reflected waves will be in phase, whereas when $n_2 \cos\theta_I < n_1 \cos\theta_T$ they will be out of phase by π. The incident and reflected waves are always in phase.

There is no Brewster angle for perpendicular polarization, since it would require:

$$\frac{n_1\mu_2}{n_2\mu_1} = \frac{\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_B}}{\cos\theta_B} \tag{10.8.43}$$

$$\iff \frac{n_1^2}{n_2^2}\frac{\mu_2^2}{\mu_1^2} = 1 \tag{10.8.44}$$

Since $\mu_1 \approx \mu_2$ this can only be true if $n_1 = n_2$, as desired.

Polarization by reflection

Consider an arbitrary wave incident on a medium interface at an angle close to the Brewster angle. The polarization of this wave may be decomposed into a component perpendicular to the plane of incidence, and a component parallel to the plane of incidence.

The reflected wave with polarization parallel to the plane of incidence vanishes at angles close to the Brewster angle $\theta_B \approx \tan^{-1} \frac{n_2}{n_1}$. Consequently when looking at the reflected wave, it will only have a polarization perpendicular to the plane of incidence. It will therefore be linearly polarized. This can be observed on a daily basis when looking at the reflection on a windowpane with polaroid glasses, or more meticulously with a polarizing filter.

This is why we often insert polarizing filters on DSLR cameras, they help reduce the glare off reflecting surfaces. Unfortunately the situation is more complicated with metals which have a different scattering behaviour.

Figure 10.6. Effect of adding a linear polarizing filter to a camera

10.9 Absorption and dispersion in dielectrics

We now seek to develop a theory on the index of refraction for materials. We may view refraction as a scattering event, where an incoming electromagnetic wave excites the bound electrons in a dielectric, inducing an oscillating dipole moment which produces radiating fields which are phase shifted from the original fields. The resulting wave will therefore be a phase shifted version of the original wave, which we may interpret as having a different phase velocity.

For a rarefied gas, we may ignore the radiating fields from other sources when writing down the equations of motion of one individual atom. Hence, modelling the electron as bound to the atom by a restoring potential which we approximate to a harmonic oscillator $U(x) = \frac{1}{2}m\omega_0^2 x^2$, with a damping force $\omega\dot{x}$, then the equation of motion reads:

$$F = m\ddot{x} = qE - m(\gamma\dot{x} + \omega_0^2 x) \implies \ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{qE}{m}e^{i\omega t} \tag{10.9.1}$$

where $\widetilde{\mathbf{E}} = Ee^{i\omega t}\hat{\mathbf{x}}$ is the impinging electric field on the atom. In the transient state solution, the solution will oscillate at the driving frequency, with different amplitude. We therefore substitute the ansatz $x = x_0 e^{i\omega t}$ into (10.9.1):

$$(-\omega^2 + i\omega\gamma + \omega_0^2)x_0 e^{i\omega t} = \frac{qE}{m}e^{i\omega t} \tag{10.9.2}$$

$$\implies x_0 = \frac{qE/m}{-\omega^2 + i\omega\gamma + \omega_0^2} \tag{10.9.3}$$

so that:

$$x = \frac{q/m}{-\omega^2 + i\omega\gamma + \omega_0^2}Ee^{i\omega t} \tag{10.9.4}$$

or alternatively:

$$\widetilde{\mathbf{p}} = \frac{q/m}{-\omega^2 + i\omega\gamma + \omega_0^2}\widetilde{\mathbf{E}} \tag{10.9.5}$$

where $\widetilde{\mathbf{p}}$ is the complex dipole moment. Recalling that the atomic polarizability is defined as satisfying $\mathbf{p} = \alpha(\omega)\mathbf{E}$ we may therefore analogously define the complex atomic polarizability $\widetilde{\alpha}$ so that:

$$\boxed{\widetilde{\alpha}(\omega) = \frac{q^2/m}{-\omega^2 + i\omega\gamma + \omega_0^2}} \tag{10.9.6}$$

Now let us model the rarefied gas as a collection of N molecules per unit volume, each with n_i electrons which have natural frequency ω_i and damping coefficient γ_i. Then the total polarizability becomes:

$$\widetilde{\alpha}(\omega) = \frac{Nq^2}{m}\sum_i \frac{n_i}{-\omega^2 + i\omega\gamma_i + \omega_i^2} \tag{10.9.7}$$

The complex susceptibility $\widetilde{\chi}_e$ satisfies $\widetilde{\mathbf{P}} = \varepsilon_0\widetilde{\chi}_e\widetilde{\mathbf{E}} = \widetilde{\alpha}\widetilde{\mathbf{E}}$ and hence:

$$\widetilde{\chi}_e = \frac{\widetilde{\alpha}}{\varepsilon_0} = \frac{Nq^2}{\varepsilon_0 m}\sum_i \frac{n_i}{-\omega^2 + i\omega\gamma_i + \omega_i^2} \tag{10.9.8}$$

Similarly we find the complex relative permittivity $\varepsilon_r = 1 + \widetilde{\chi}_e$ to be:

$$\boxed{\widetilde{\varepsilon}_r = 1 + \frac{Nq^2}{\varepsilon_0 m} \sum_i \frac{n_i}{-\omega^2 + i\omega\gamma_i + \omega_i^2}} \tag{10.9.9}$$

Now the complex index of refraction for a dielectric medium (where we may take $\mu_r = 1$) is:

$$\tilde{n}^2 = \widetilde{\varepsilon}_r = 1 + \frac{Nq^2}{\varepsilon_0 m} \sum_i \frac{n_i}{-\omega^2 + i\omega\gamma_i + \omega_i^2} \tag{10.9.10}$$

For large N we may regard $n = \sqrt{1+\epsilon} \approx 1 + \frac{1}{2}\epsilon$ and hence:

$$\tilde{n} \approx 1 + \frac{Nq^2}{2\varepsilon_0 m} \sum_i \frac{n_i}{-\omega^2 + i\omega\gamma_i + \omega_i^2} \tag{10.9.11}$$

We may then take the real part of this expression to find that the real index of refraction is:

$$\boxed{n \approx 1 + \frac{Nq^2}{2\varepsilon_0 m} \sum_i \frac{n_i(\omega_i^2 - \omega)^2}{(\omega_i^2 - \omega^2)^2 + \gamma_i^2\omega^2}} \tag{10.9.12}$$

whereas the imaginary component is:

$$n_I \approx -\frac{Nq^2}{2\varepsilon_0 m} \sum_i \frac{n_i \omega \gamma_i}{(\omega_i^2 - \omega^2)^2 + \gamma_i^2\omega^2} \tag{10.9.13}$$

We have failed to account for the radiating electromagnetic fields due to the neighbouring atoms. Let us consider a spherical surface enclosing and centered on the atom we've been considering. All atoms outside this surface will be far enough so that the dipole approximation will be sufficient to describe the electric fields they produce. We denote this electric field as $\widetilde{E}_{else}$. The average field inside this surface is instead the field due to a uniformly polarized sphere due to $\widetilde{E}_{else}$, that is $-\frac{\widetilde{P}}{3\varepsilon_0} = -\frac{\widetilde{\alpha}\widetilde{E}_{else}}{3\varepsilon_0}$ so the total field will actually be:

$$\widetilde{E} = \widetilde{E}_{else} - \frac{\widetilde{P}}{3\varepsilon_0} = \left(1 - \frac{\widetilde{\alpha}}{3\varepsilon_0}\right)\widetilde{E}_{else} \tag{10.9.14}$$

Consequently we find that:

$$\widetilde{P} = \varepsilon_0 \widetilde{\chi}_e \widetilde{E} = \varepsilon_0 \widetilde{\chi}_e \left(1 - \frac{\widetilde{\alpha}}{3\varepsilon_0}\right)\widetilde{E}_{else} = \widetilde{\alpha}\widetilde{E}_{else} \tag{10.9.15}$$

implying that:

$$\widetilde{\chi}_e = \frac{\widetilde{\alpha}/\varepsilon_0}{1 - \widetilde{\alpha}/3\varepsilon_0} \tag{10.9.16}$$

and thus:

$$\widetilde{n}^2 = \widetilde{\varepsilon}_r = 1 + \frac{\widetilde{\alpha}/\varepsilon_0}{1 - \widetilde{\alpha}/3\varepsilon_0} \tag{10.9.17}$$

Rearranging we find that:

$$\boxed{\widetilde{\alpha} = 3\varepsilon_0 \frac{\widetilde{n}^2 - 1}{\widetilde{n}^2 + 2} = \frac{Nq^2}{m} \sum_i \frac{n_i}{-\omega^2 + i\omega\gamma_i + \omega_i^2}} \tag{10.9.18}$$

To find the real and complex parts of $\hat{n} = n_R + in_I$ we need to perform some more algebra. Starting from with $\widetilde{n} = n_R + in_I$ and $\widetilde{\varepsilon}_r = \varepsilon_{r,R} + i\varepsilon_{r,I}$:

$$n_R^2 + 2in_Rn_I - n_I^2 = \varepsilon_{r,R} + i\varepsilon_{r,I} \implies \begin{cases} n_R^2 - n_I^2 = \varepsilon_{r,R} \\ 2n_Rn_I = \varepsilon_{r,I} \end{cases} \tag{10.9.19}$$

which yields

$$n_R^4 - \varepsilon_{r,R}n_R^2 - \frac{\varepsilon_{r,i}}{4} = 0 \tag{10.9.20}$$

The solution is therefore:

$$\boxed{\begin{cases} n_R^2 = \frac{\varepsilon_{r,R} + \sqrt{\varepsilon_{r,R} + \varepsilon_{r,I}}}{2} \\ n_I^2 = \frac{-\varepsilon_{r,R} + \sqrt{\varepsilon_{r,R} + \varepsilon_{r,I}}}{2} \end{cases}} \tag{10.9.21}$$

Absorption

The fact that the index of refraction n, and consequently the wavenumber $k = \frac{\omega}{c}n$, may be expressed as complex numbers means that the plane wave-solutions to the wave equation won't always be sinusoidal.

We recall that for LIH dispersive materials, where $\widetilde{\mathbf{D}} = \widetilde{\varepsilon}\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}} = \frac{1}{\mu_0}\widetilde{\mathbf{B}}$, we find:

$$\nabla \cdot \widetilde{\mathbf{E}} = 0 \tag{10.9.22}$$

$$\nabla \cdot \widetilde{\mathbf{B}} = 0 \tag{10.9.23}$$

$$\nabla \times \widetilde{\mathbf{E}} = -\frac{\partial \widetilde{\mathbf{B}}}{\partial t} \tag{10.9.24}$$

$$\nabla \times \widetilde{\mathbf{B}} = \mu_0 \widetilde{\varepsilon} \frac{\partial \mathbf{E}}{\partial t} \tag{10.9.25}$$

Let us now see what plane wave solutions of the form $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{B}}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$. We find that:

$$\nabla \cdot \mathbf{E} = 0 \implies \mathbf{k} \cdot \widetilde{\mathbf{E}}_0 = 0 \tag{10.9.26}$$

$$\nabla \cdot \mathbf{B} = 0 \implies \mathbf{k} \cdot \widetilde{\mathbf{B}}_0 = 0 \tag{10.9.27}$$

$$\tag{10.9.28}$$

so the waves are going to be transverse.

Furthermore:

$$\nabla \times \mathbf{E} = -\frac{\partial \widetilde{\mathbf{B}}}{\partial t} \implies \mathbf{k} \times \widetilde{\mathbf{E}}_0 = \omega \widetilde{\mathbf{B}}_0 \tag{10.9.29}$$

$$\nabla \times \mathbf{B} = \mu_0 \widetilde{\varepsilon} \frac{\partial \widetilde{\mathbf{E}}}{\partial t} \implies \mathbf{k} \times \widetilde{\mathbf{B}}_0 = -\mu_0 \widetilde{\varepsilon} \omega^2 \widetilde{\mathbf{E}}_0 \tag{10.9.30}$$

which we combine to find that:

$$\mathbf{k} \times (\mathbf{k} \times \widetilde{\mathbf{E}}_0) = -\mathbf{k}^2 \widetilde{\mathbf{E}}_0 = -\mu_0 \widetilde{\varepsilon} \omega \widetilde{\mathbf{E}}_0 \tag{10.9.31}$$

$$\mathbf{k}^2 = \mu_0 \widetilde{\varepsilon} \omega^2 \tag{10.9.32}$$

so we see that $\mathbf{k}$ will actually be complex. Changing our notation accordingly:

$$\widetilde{\mathbf{k}}^2 = \mu_0 \widetilde{\varepsilon} \omega^2 = \frac{\omega^2}{\widetilde{v}^2} \implies \widetilde{k} = \frac{\omega}{\widetilde{v}} = \frac{c}{\widetilde{v}} \frac{\omega}{c} = \widetilde{n} \frac{\omega}{c} \tag{10.9.33}$$

Since $\widetilde{n} = \sqrt{\widetilde{\varepsilon}_r}$ for non-magnetic materials we therefore find that:

$$\widetilde{k} = \sqrt{\widetilde{\varepsilon}_r} \frac{\omega}{c} \tag{10.9.34}$$

Inserting $\widetilde{\mathbf{k}} = \mathbf{k}_R + i\mathbf{k}_I$ into the plane wave ansatz we find that:

$$\boxed{\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 e^{-\mathbf{k}_I \cdot \mathbf{r}} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}} \tag{10.9.35}$$

and similarly:

$$\boxed{\widetilde{\mathbf{B}} = \widetilde{\mathbf{B}}_0 e^{-\mathbf{k}_I \cdot \mathbf{r}} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}} \tag{10.9.36}$$

We get an exponential attenuation due to the dielectric absorbing part of the electromagnetic field's energy. The direction of attenuation is in the propagation direction $\mathbf{k}$

We define the length $L_{abs} = \frac{1}{k_I}$ as the absorption length, the distance over which the electric field is reduced by $e^{-1} \approx 0.37$.

Suppose $\widetilde{E}_0 = E_0 e^{i\delta}$, then we have that:

$$\widetilde{B}_0 = \frac{\widetilde{k}}{\omega} \widetilde{E}_0 = \frac{|\widetilde{k}|}{\omega} e^{i\phi} e^{i\delta}, \ \phi = \arctan \frac{k_I}{k_R} \tag{10.9.37}$$

so the magnetic field will lag behind the electric field by, and will be reduced by a factor of:

$$\frac{B_0}{E_0} \frac{|\widetilde{k}|}{\omega} = \frac{|\widetilde{n}|}{c} = \frac{1}{c} (\varepsilon_{r,R} + \varepsilon_{r,I})^{1/4} \tag{10.9.38}$$

Dispersion and rainbows

These equations relating the phase velocity $v = \frac{c}{n}$ and the angular frequency ω of a wave are known as **dispersion relations** when travelling through **dispersive media**. They explain why light of different frequencies will refract at different angles due to their different speeds.

The phenomenon of dispersion explains several physical phenomena, most notably the formation of rainbows.

Let us consider a dispersive medium composed of several spherical droplets of water of radius a.

We consider a ray of sunlight hitting one of these droplets at a height y to its center, and hence at an angle $\theta_i = \sin^{-1}\frac{y}{a}$. Its transmitted component will be refracted at an angle θ_t satisfying $\sin\theta_i = n(\omega)\sin\theta_t$ where $n(\omega)$ is the (frequency dependent) index of refraction for water. The associated deviation is $\theta_i - \theta_t$. The ray will then hit the back of the droplet, and its reflected component will be reflected back at the same angle θ_t. The deviation for this event is $\pi - 2\theta_t$ Finally the ray will ht the lower part of the droplet, and get transmitted at an angle θ_i. The associated deviation is again $\theta_i - \theta_t$.

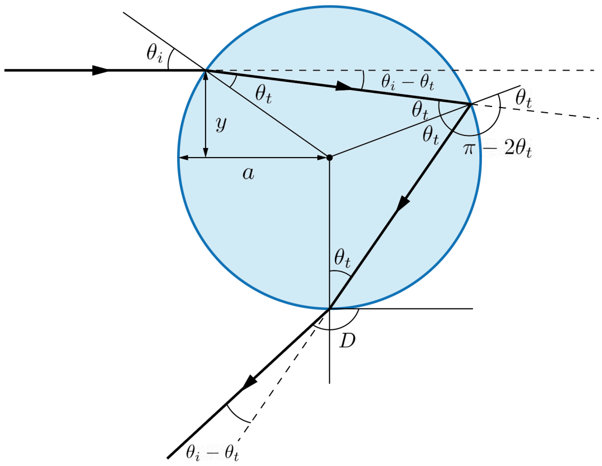


Figure 10.7. Deviation of light ray by water droplet

The total deviation D of the light ray must then be:

$$D = (\theta_i - \theta_t) + (180^\circ - 2\theta_t) + (\theta_i - \theta_t) = 180^\circ + 2\theta_i - 4\theta_t \tag{10.9.39}$$

Since $\theta_i = \sin^{-1}\frac{y}{a}$ and $\theta_t = \sin^{-1}\frac{y}{n(\omega)a}$ we get the following relation for D in terms of $x = \frac{y}{a}$:

$$\boxed{D(x) = 180^\circ + 2\sin^{-1}x - 4\sin^{-1}\left(\frac{x}{n(\omega)}\right)} \tag{10.9.40}$$

This may be plotted for red light ($n_r \approx 1.331$), green light ($n_G \approx 1.336$) and blue light ($n_B \approx 1.340$) as shown below:

Firstly, note that the deviation is minimized to some value $D_{\text{min}}(\omega)$, which for red light is around $137.6°$, and which increases for other colors. Monochromatic light of frequency ω

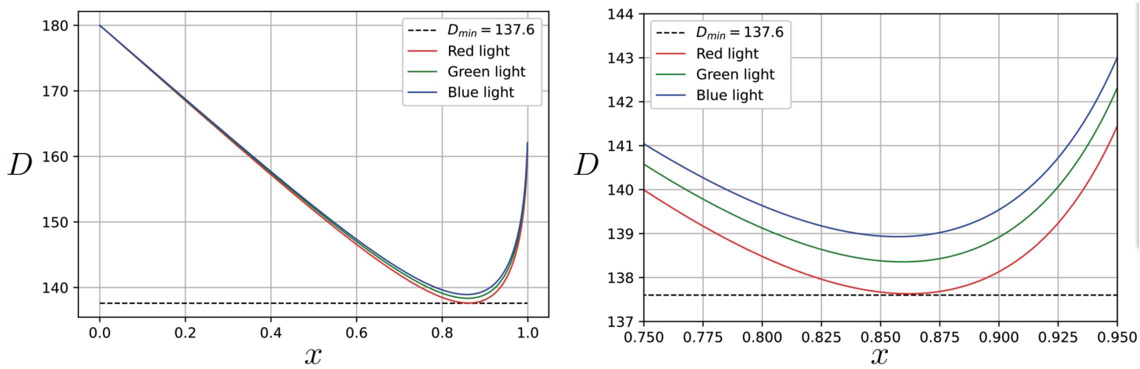


Figure 10.8. Plots of deviation angles for light rays of different frequencies

deviated by $D_{\min}(\omega)$ a very high intensity since light incident at slightly different x will still get the same deviation to a good approximation ($\left.\frac{\partial D(\omega,x)}{\partial x}\right|_{min} = 0$).

We see that red light has the smallest angular deviation, whereas blue light has the largest deviation. Hence, when a light ray hits a droplet it will get dispersed as shown below:

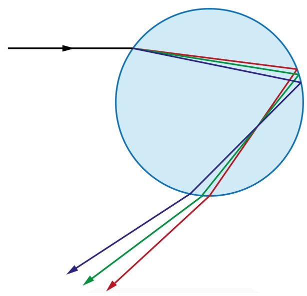


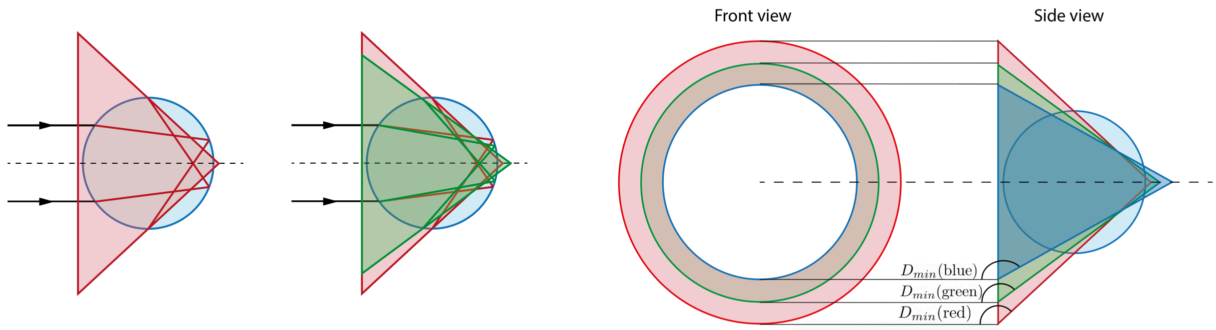
Figure 10.9. Dispersion of light by a water droplet

If we consider an array of parallel red light rays hitting the droplet, then the deviation D will range between $D_{\min}(\text{red})$ and $180°$, so the scattered light will form a cone as shown below:

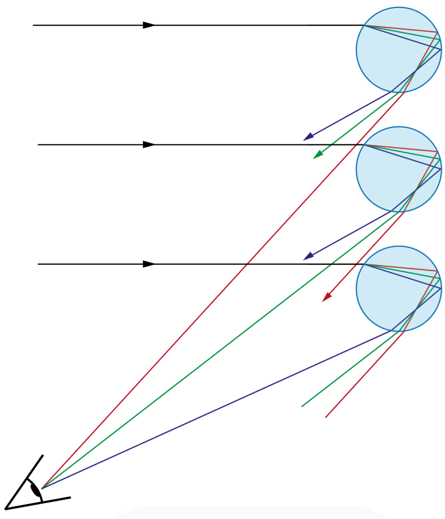
Outside of the cones no light will get scattered, whereas any region within the cone will get some sort of light.

If we include green light then we get another cone. If we include all frequencies in the visible range then we will end up with the superposition of infinitely many cones. For angles larger than $D_{\min}(\text{red})$ no light can get scattered, whereas for angles smaller than $D_{\min}(\text{blue})$ all frequencies get mixed, giving white light.

Consequently, if a bystander looks up to a droplet at an angle of $180° - D_{\min}(\text{red}) = 42.4°$, they will only receive the red component of the light. For droplets at at a higher angle, no light (in the visible spectrum) will be capable to reach the observer since they get scattered



at smaller angles (they subtend a smaller cone). Instead, droplets at a lower angle, say D_{min}(yellow), the yellow component will dominate as it will get reflected more strongly. Looking even more down the green will dominate, and so forth until the blue component dominates. For droplets that are even lower, we won't have colors in the visible spectrum which are aligned at an angle $D_{\text{min}}(\omega)$. Therefore all colors will be reaching the observer, which would see white light (technically it would be sky-blue due to the scattering of light by the atmosphere, as we will explain in the chapter on radiation).



Hence when looking at a rainbow, each droplet will mainly scatter only one frequency onto the observer, this color will be the one whose minimum deviation D_{min} matches exactly with the angle the observer makes with the droplet. This explains why red is at the top, and blue is at the bottom.

However, we haven't addressed the issue of the shape of a rainbow, which is semi-circular usually. This can be explained simply by noting that the locus of points that subtend an angle $180° - D_{\text{min}}$ forms a semi-circle in our field of view (the ground blocks the lower semi-circle). The angle subtended by the rainbow is then $D_{\text{min}}(\text{blue}) - D_{\text{min}}(\text{red}) \approx 2°$.

Group vs phase velocity

We saw that the phase velocity describes the velocity at which a monochromatic wave (single-frequency ω) travels:

$$v_p = \frac{\omega}{k} \tag{10.9.41}$$

Suppose we have a superposition of several waves with different frequencies, a pulse given by:

$$\widetilde{\mathbf{E}}(\mathbf{r}) = \frac{1}{2\pi}\int \widetilde{\mathbf{E}}(k)e^{i(kz-\omega t)}dk \tag{10.9.42}$$

If $\mathbf{E}(k)$ has a sharp peak at some k_0, then we may perform a taylor expansion:

$$kz - \omega t \approx kz - \omega(k_0)t - \frac{d\omega}{dk}\bigg|_{k_0}(k-k_0)t \tag{10.9.43}$$

$$= -\left(\omega(k_0) - k_0\frac{d\omega}{dk}\bigg|_{k_0}\right)t + \left(z - \frac{d\omega}{dk}\bigg|_{k_0}t\right)k \tag{10.9.44}$$

We identify the group velocity $v_g(k_0) = \frac{d\omega}{dk}\big|_{k_0}$ and hence:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \frac{e^{i[k_0 v_g(k_0)-\omega(k_0)]t}}{2\pi}\int \widetilde{\mathbf{E}}(k)e^{ik(z-v_g(k_0)t)}dk \tag{10.9.45}$$

The peak of the pulse ($k = k_0$) therefore travels at the group velocity v_g. Consequently, using $k = n(\omega)\frac{\omega}{c}$ we find that:

$$\frac{dk}{d\omega} = \frac{dn}{d\omega}\frac{\omega}{c} + \frac{n(\omega)}{c} = \frac{dn}{d\omega}\frac{\omega}{c} + \frac{1}{v_p} \tag{10.9.46}$$

10.10 Absorption and dispersion by conductors

Our treatment of electromagnetic waves in matter has thus far assumed that there are no free charges nor currents in the propagation medium. The story changes however when we consider conductors, where to a good approximation $\mathbf{J}_f = \sigma\mathbf{E}$ and $\rho_b = 0$.

Taking the divergence and using Gauss' law together with the continuity equation:

$$\nabla \cdot \mathbf{J}_f = \sigma\nabla \times \mathbf{E} = \frac{\sigma}{\varepsilon}\rho_f = -\frac{\partial \rho_f}{\partial t} \tag{10.10.1}$$

giving us:

$$\rho_f(t) = \rho_f(0)e^{-t/\tau},\ \tau = \frac{\varepsilon}{\sigma} \tag{10.10.2}$$

Therefore, if we place some charges ρ_f within a conductor, they will quickly move to the surface in characteristic time $\tau = \frac{\varepsilon_0}{\sigma}$. The better the conductor, the larger σ and thus the less time it takes for this to occur. A perfect conductor satisfies $\sigma \to \infty$ and thus the charge will instantaneously move to the surface.

Suppose we wait long enough for $\rho_f = 0$ to a good approximation. Then Maxwell's equations read:

$$\nabla \cdot \widetilde{\mathbf{E}} = 0 \tag{10.10.3}$$

$$\nabla \cdot \widetilde{\mathbf{B}} = 0 \tag{10.10.4}$$

$$\nabla \times \widetilde{\mathbf{E}} = -\frac{\partial \widetilde{\mathbf{B}}}{\partial t} \tag{10.10.5}$$

$$\nabla \times \widetilde{\mathbf{B}} = \mu\varepsilon\left(\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\tau}\mathbf{E}\right) \tag{10.10.6}$$

giving us a modified wave equation:

$$\boxed{\nabla^2 \mathbf{E} = \frac{1}{v^2}\left(\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\tau}\frac{\partial \mathbf{E}}{\partial t}\right)} \tag{10.10.7}$$

Let us see what plane wave solutions are admitted by substituting the typical ansatz:

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \ \widetilde{\mathbf{B}} = \widetilde{\mathbf{B}}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10.10.8}$$

into (10.10.7). We find that

$$-k^2 \widetilde{\mathbf{E}} = \frac{1}{v^2}\left(-\omega^2 - \frac{i\omega}{\tau}\right)\widetilde{\mathbf{E}} \implies k^2 = \frac{\omega^2}{v^2} + \frac{i\omega}{\tau v^2} \tag{10.10.9}$$

We therefore expect that k will be a complex number and may be written as $\widetilde{k} = k_R + ik_I$. Then:

$$\begin{cases} k_R^2 - k_I^2 = \frac{\omega^2}{v^2} \\ 2k_R k_I = \frac{\omega}{\tau v^2} \end{cases} \implies k_R^4 - \frac{\omega^2}{v^2}k_R^2 - \left(\frac{\omega}{2\tau v^2}\right)^2 = 0 \tag{10.10.10}$$

Hence:

$$k_R^2 = \frac{\omega^2}{2v^2}\left(\sqrt{1 + \frac{1}{\tau^2\omega^2}} + 1\right), \ k_I^2 = \frac{\omega^2}{2v^2}\left(\sqrt{1 + \frac{1}{\tau^2\omega^2}} - 1\right) \tag{10.10.11}$$

or:

$$\boxed{k_R = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \frac{\sigma^2}{\varepsilon^2\omega^2}} + 1\right]^{1/2}} \tag{10.10.12a}$$

$$\boxed{k_I = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \frac{\sigma^2}{\varepsilon^2\omega^2}} - 1\right]^{1/2}} \tag{10.10.12b}$$

Consequently, as in the case of dispersive media, the electromagnetic field will be attenuated:

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 e^{-\mathbf{k}_I\cdot\mathbf{r}} e^{i(\mathbf{k}_R\cdot\mathbf{r} - \omega t)} \tag{10.10.13}$$

$$\widetilde{\mathbf{B}} = \widetilde{\mathbf{B}}_0 e^{-\mathbf{k}_I\cdot\mathbf{r}} e^{i(\mathbf{k}_R\cdot\mathbf{r} - \omega t)} \tag{10.10.14}$$

We define the skin depth to be $d = \frac{1}{k_I}$ or:

$$d = \frac{2\tau c^2}{\omega}\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}\right]^{1/2} \tag{10.10.15}$$

$$= \frac{1}{\sigma}\sqrt{\frac{2\varepsilon}{\mu}}\left[1+\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}\right]^{1/2} \tag{10.10.16}$$

For poor conductors where $\sigma \ll \omega\varepsilon$ then:

$$\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}} \approx 1+\frac{\sigma^2}{2\varepsilon^2\omega^2} \implies \left[1+\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}\right]^{1/2} \approx \sqrt{2+\frac{\sigma^2}{2\varepsilon^2\omega^2}} \approx \sqrt{2} \tag{10.10.17}$$

so that $d \approx \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$.

Finally, recall that:

$$\widetilde{\mathbf{k}} \times \widetilde{\mathbf{E}}_0 = \omega\widetilde{\mathbf{B}}_0 \tag{10.10.18}$$

and therefore:

$$\widetilde{\mathbf{B}} = \frac{\widetilde{\mathbf{k}} \times \widetilde{\mathbf{E}}_0}{\omega} e^{-\mathbf{k}_I\cdot\mathbf{r}} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} \tag{10.10.19}$$

Hence if $\widetilde{\mathbf{E}}_0 = E_0 e^{i\delta}\hat{\mathbf{x}}$ and $\widetilde{\mathbf{k}} = \hat{\mathbf{y}}$ then $\widetilde{\mathbf{B}}_0 = \frac{\widetilde{k}}{\omega}E_0 e^{i\delta}(\hat{\mathbf{y}}\times\hat{\mathbf{x}}) = -\frac{|\widetilde{k}|}{\omega}E_0 e^{i(\delta+\phi)}\hat{\mathbf{z}}$ where $\phi = \arctan\frac{k_I}{k_R}$. The magnetic field therefore lags behind the electric field by $\arctan\frac{k_I}{k_R}$ and is reduced by a factor of $\frac{|\widetilde{k}|}{\omega}$ which we may simplify further:

$$\frac{B_0}{E_0} = \frac{|\widetilde{k}|}{\omega} = \sqrt{\frac{\varepsilon\mu}{2}\left(1+\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}-1+\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}\right)} \tag{10.10.20}$$

$$= \sqrt{\varepsilon\mu\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}} \tag{10.10.21}$$

Reflection by perfect conductors

For a perfect conductor, the electric and magnetic field decay immediately due to a vanishing skin depth. Therefore, $E^{\parallel}$ and $B^{\perp}$ must be zero just outside the surface in order to satisfy the relevant boundary conditions.

Suppose then that we have a plane wave incident on a perfect conductor at an angle θ_i. we set the z axis so that the conductor lies in the $z=0$ plane, while the $x=0$ plane forms the scattering plane.

Then:

$$\widetilde{\mathbf{E}}_I = E_{I0}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\mathbf{x} \tag{10.10.22}$$

The reflected wave is then:

$$\widetilde{\mathbf{E}}_R = E_{R0} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \mathbf{x} \tag{10.10.23}$$

while the transmitted wave is of course zero.

From the boundary conditions we see that:

$$E_{I0} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} + E_{R0} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} = 0 \tag{10.10.24}$$

so once again we recover Snell's laws $\theta_I = \theta_R$. Then writing:

$$\mathbf{k}_I \cdot \mathbf{r} = k_0 y \sin\theta_I + k_0 z \sin\theta_I \tag{10.10.25}$$

$$\mathbf{k}_R \cdot \mathbf{r} = k_0 y \sin\theta_I - k_0 z \sin\theta_I \tag{10.10.26}$$

we find that at $z = 0$:

$$E_{I0} e^{i(k_0 y \sin\theta_I - \omega t)} + E_{R0} e^{i(k_0 y \sin\theta_I - \omega t)} = 0 \tag{10.10.27}$$

implying that $E_{I0} = -E_{R0}$, there is a π phase shift between the incident and reflected waves. These results coincide with (**??**) taking $\beta = 0$ in the perfect conductor limit.

The total electric field is:

$$\widetilde{\mathbf{E}} = \widetilde{E}_0 (e^{i(k_0 y \sin\theta_I - \omega t)} - e^{i(k_0 y \sin\theta_I - \omega t)}) \mathbf{x} \tag{10.10.28}$$

$$= 2i\widetilde{E}_0 \sin(k_0 z \cos\theta) e^{i(k_0 y \sin\theta - \omega t)} \mathbf{x} \tag{10.10.29}$$

10.11 Waveguides

Suppose we have a hollow pipe, known as a **waveguide**, of size $0 \leq x \leq a$, $0 \leq y \leq b$ made out of a perfectly conducting material, so that:

$$\mathbf{E}^{||} = 0,\ \mathbf{B}^{\perp} = 0,\ x = 0, a,\ y = 0, b \tag{10.11.1}$$

at the inner wall. Let's consider waves travelling down the pipe:

$$\widetilde{\mathbf{E}}(x, y, z, t) = \widetilde{\mathbf{E}}_0(x, y) e^{i(kz - \omega t)} \tag{10.11.2}$$

$$\widetilde{\mathbf{B}}(x, y, z, t) = \widetilde{\mathbf{B}}_0(x, y) e^{i(kz - \omega t)} \tag{10.11.3}$$

Since these waves are not plane waves, they won't be transverse in general, so $\mathbf{E}_0$ and $\mathbf{B}_0$ will have z-components as well:

$$\widetilde{\mathbf{E}}_0 = E_x \mathbf{x} + E_y \mathbf{y} + E_z \mathbf{z} \tag{10.11.4}$$

$$\widetilde{\mathbf{B}}_0 = B_x \mathbf{x} + B_y \mathbf{y} + B_z \mathbf{z} \tag{10.11.5}$$

$$\tag{10.11.6}$$

We can substitute these into Maxwell's equations in vacuum. For example, Faraday's law gives:

$$(\nabla \times \mathbf{E})_x = -\frac{\partial \widetilde{\mathbf{B}}_x}{\partial t} \implies \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x \tag{10.11.7}$$

$$(\nabla \times \mathbf{E})_y = -\frac{\partial \widetilde{\mathbf{B}}_y}{\partial t} \implies ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y \tag{10.11.8}$$

$$(\nabla \times \mathbf{E})_z = -\frac{\partial \widetilde{\mathbf{B}}_z}{\partial t} \implies \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \tag{10.11.9}$$

$$\tag{10.11.10}$$

Similarly, the Ampere-Maxwell law gives:

$$(\nabla \times \mathbf{B})_x = \frac{1}{c^2}\frac{\partial \widetilde{\mathbf{E}}_x}{\partial t} \implies \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x \tag{10.11.11}$$

$$(\nabla \times \mathbf{B})_y = \frac{1}{c^2}\frac{\partial \widetilde{\mathbf{E}}_y}{\partial t} \implies ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \tag{10.11.12}$$

$$(\nabla \times \mathbf{B})_z = \frac{1}{c^2}\frac{\partial \widetilde{\mathbf{E}}_z}{\partial t} \implies \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \tag{10.11.13}$$

$$\tag{10.11.14}$$

Firstly, we find that:

$$E_y = -\frac{\omega}{k}B_x - \frac{i}{k}\frac{\partial E_z}{\partial y} \tag{10.11.15}$$

and similarly:

$$B_x = -\frac{i}{k}\frac{\partial B_z}{\partial x} - \frac{\omega}{kc^2}E_y \tag{10.11.16}$$

Note that since the solutions we are considering generally won't be plane waves, we can't use $\omega = kc$ to simplify our results.

Substituting (10.11.22) into (10.11.19) then:

$$E_y = \frac{\omega}{k}\left(\frac{i}{k}\frac{\partial B_z}{\partial x} + \frac{\omega}{kc^2}E_y\right) - \frac{i}{k}\frac{\partial E_z}{\partial y} \tag{10.11.17}$$

$$\iff E_y\left(k^2 - \frac{\omega^2}{c^2}\right) = i\left(\omega\frac{\partial B_z}{\partial x} - k\frac{\partial E_z}{\partial y}\right) \tag{10.11.18}$$

$$\iff \boxed{E_y = \frac{i}{k^2 - (\omega/c)^2}\left(\omega\frac{\partial B_z}{\partial x} - k\frac{\partial E_z}{\partial y}\right)} \tag{10.11.19}$$

Similarly we also find that:

$$E_x = -\frac{i}{k^2 - (\omega/c)^2}\left(\omega\frac{\partial B_z}{\partial y} + k\frac{\partial E_z}{\partial x}\right) \tag{10.11.20}$$

$$B_y = -\frac{i}{k^2 - (\omega/c)^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right) \tag{10.11.21}$$

$$B_x = -\frac{i}{k^2 - (\omega/c)^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \tag{10.11.22}$$

Now using Gauss' Law:

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0 \tag{10.11.23}$$

and since:

$$\frac{\partial E_x}{\partial x} = -\frac{i}{k^2 - (\omega/c)^2}\left(\omega\frac{\partial^2 B_z}{\partial y \partial x} + k\frac{\partial^2 E_z}{\partial x^2}\right) \tag{10.11.24}$$

$$\frac{\partial E_y}{\partial y} = \frac{i}{k^2 - (\omega/c)^2}\left(\omega\frac{\partial^2 B_z}{\partial x \partial y} - k\frac{\partial^2 E_z}{\partial y^2}\right) \tag{10.11.25}$$

we find that

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{ik}{k^2 - (\omega/c)^2}\left(\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)E_z \tag{10.11.26}$$

Gauss' Law thus reduces to:

$$\boxed{\left[\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right]E_z = 0} \tag{10.11.27}$$

Similarly using the no-monopole law one finds that:

$$\boxed{\left[\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right]B_z = 0} \tag{10.11.28}$$

We can use these equations to find E_z and B_z, and then use (10.11.19),(10.11.20), (10.11.21), (10.11.22) to find E_y, E_x, B_y, B_x and thus $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$.

There are special solutions to these problems by setting E_z, B_z to zero:

(i) If $E_z = 0$ then we call these solutions TE waves.

(ii) If $B_z = 0$ then we call these solutions TM waves.

(iii) If $E_z = B_z = 0$ then we call these solutions TEM waves.

TE waves

Let $E_z = 0$, and $B_z = X(x)Y(y)$. Then substituting into (10.11.28):

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right) \tag{10.11.29}$$

implying that:

$$\frac{\partial^2 X}{\partial x^2} = -k_x^2 X,\ \frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y,\ k_x^2 + k_y^2 = \frac{\omega^2}{c^2} - k^2 \tag{10.11.30}$$

The general solutions are given by:

$$X = A\sin k_x x + B\cos k_x x,\ Y = C\sin k_y y + D\cos k_y y \tag{10.11.31}$$

Now let's use the boundary conditions:

$$y = 0, b:\ B_y = 0, E_{x,z} = 0 \tag{10.11.32}$$

$$x = 0, a:\ B_x = 0,\ E_{y,z} = 0 \tag{10.11.33}$$

Firstly, $B_x \propto \frac{\partial B_z}{\partial x} \propto \frac{\partial X}{\partial x} = 0$ at $x = 0, a$ so:

$$k_x(A\cos k_x x - B\sin k_x x) = 0,\ x = 0, a \tag{10.11.34}$$

from which we deduce that $A = 0$ and $k_x = \frac{n\pi}{a}$ where $n = 0, 1, \ldots$. $E_z = 0$ has already been satisfied, while $E_x = 0$ reduces to $\frac{\partial B_z}{\partial x} = 0$.

Similarly $C = 0$ and $k_y = \frac{m\pi}{b}$. The final solution is:

$$\boxed{B_z = B_0 \cos\frac{m\pi y}{b}\cos\frac{n\pi x}{a}} \tag{10.11.35}$$

which is known as the TE_{mn} mode. Its wavenumber is:

$$k = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2} \tag{10.11.36}$$

Let us define the **cutoff frequency** ω_{nm} as:

$$\boxed{\omega_{nm} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}} \tag{10.11.37}$$

then we rewrite the wavenumber as:

$$\boxed{k = \frac{1}{c}\sqrt{\omega^2 - \omega_{nm}^2}} \tag{10.11.38}$$

Note that if $\omega < \omega_{nm}$, then the wavenumber will be purely imaginary, and hence the wave will be attenuated, decaying exponentially with decay length $\frac{c}{\sqrt{\omega_{nm}^2-\omega^2}}$. This is why ω_{nm} is known as the cutoff frequency.

Also, the mode TE_{00} is not possible. Indeed in this mode the solution reads $B_z = B_0$. Applying Faraday's law to the cross section of the wave-guide:

$$\int \mathbf{B}\cdot d\mathbf{a} = abB_0 = \oint \mathbf{E}\cdot d\mathbf{l} = 0 \implies B_0 = 0 \tag{10.11.39}$$

since $\mathbf{E}^{\|} = 0$ at the walls of the waveguide.

The lowest possible TE cutoff frequency, assuming $a < b$ is then ω_{01}:

$$\omega_{01} = \frac{c\pi}{b} \tag{10.11.40}$$

The phase velocity is:

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_{nm}/\omega)^2}} > c \tag{10.11.41}$$

while the group velocity is:

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \left(\frac{\omega}{c\sqrt{\omega^2-\omega_{nm}^2}}\right)^{-1} = c\sqrt{1-(\omega_{nm}/\omega)^2} < c \tag{10.11.42}$$

Luckily for us, the energy of the wave travels at the group velocity rather than the phase velocity, hence there is no contradiction with special relativity.

TM waves

The general solution is completely analogous to the TE waves, if we let $E_z = X(x)Y(y)$ then:

$$X = A\sin k_x x + B\cos k_x x,\ Y = C\sin k_y y + D\cos k_y y \tag{10.11.43}$$

The boundary conditions are still:

$$y = 0, b:\ B_y = 0, E_{x,z} = 0 \tag{10.11.44}$$

$$x = 0, a:\ B_x = 0,\ E_{y,z} = 0 \tag{10.11.45}$$

Now we find that at $x = 0, a$, $E_z = 0$ implies $B = 0$ and $k_x = \frac{n\pi}{a}$ for $n = 1, 2, ...$ Instead, $E_y = 0$ and B_x both imply $\frac{\partial E_z}{\partial y} = 0$ at $x = 0, a$, which is clearly already been satisfied.

Similarly, at $y = 0, a$ one finds that $D = 0$ and $k_y = \frac{m\pi}{b}$ for $m = 1, 2, ...$ So the general solution is:

$$\boxed{E_z = E_0 \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}} \tag{10.11.46}$$

The cut off frequency is the same as TE waves:

$$\omega_{nm} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \tag{10.11.47}$$

and hence:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{nm}^2} \tag{10.11.48}$$

Note that the lowest possible cut off requency is ω_{11}, with:

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \tag{10.11.49}$$

11 Gauge transformations

Radiation and Retarded Potentials

12

12.1 Green's functions

12.2 Retarded potentials

In electrostatics and magnetostatics we have encountered the potentials satisfying the relations (in the Lorentz gauge):

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0},\ \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{12.2.1}$$

with solutions:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}_0|}d\tau',\ \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}_0|}d\tau' \tag{12.2.2}$$

How do these equations adapt to moving sources? Since electromagnetic waves travel at the speed of light c, we have that a perturbation on the source at time t_r will only reach the observer after

$$t = t_r + \frac{|\mathbf{r}-\mathbf{r}'|}{c} \tag{12.2.3}$$

Therefore one should expect (12.2.2) to generalize to:

$$\boxed{V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\tau',\ \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\tau'} \tag{12.2.4}$$

These are known as retarded potential, as they take into account the time delay enforced by special relativity. To show that they are right, we need to show that they satisfy the Lorenz gauge conditions:

$$\nabla\cdot\mathbf{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t} \tag{12.2.5}$$

$$\Box^2 V = -\frac{\rho}{\varepsilon_0} \tag{12.2.6}$$

$$\Box^2 \mathbf{A} = -\mu_0\mathbf{J} \tag{12.2.7}$$

12.3 Electric dipole radiation

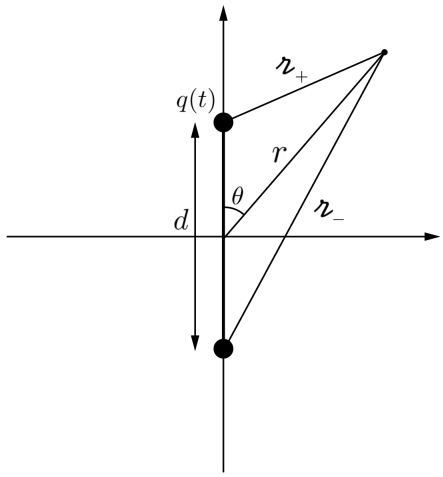


Figure 12.1. Electric dipole radiation

Consider two metal spheres a distance d apart with charge $q(t)$ and $-q(t)$ on the upper and lower spheres respectively. The spheres are connected by a metal wire which allows charge to move from one sphere to the other at a frequency ω so that:

$$q(t) = q_0 \cos \omega t \tag{12.3.1}$$

and resulting dipole:

$$\mathbf{p}(t) = p_0 \cos \omega t \hat{\mathbf{z}}, \ p_0 = q_0 d \tag{12.3.2}$$

Now the retarded potentials (12.2.4) read:

$$V(\mathbf{r}, t) = \frac{q_0}{4\pi\varepsilon_0}\left[\frac{\cos(\omega t - \omega \mathscr{r}_+/c)}{\mathscr{r}_+} - \frac{\cos(\omega t - \omega \mathscr{r}_-/c)}{\mathscr{r}_-}\right] \tag{12.3.3}$$

where $\mathscr{r}_\pm$ are the distances from the sources:

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + \frac{d^2}{4}} \implies \frac{1}{\mathscr{r}_\pm} = \frac{1}{r}\frac{1}{\sqrt{1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}} \tag{12.3.4}$$

Taking $d \ll r$ we may then approximate these distances as:

$$\mathscr{r}_\pm \approx r\left(1 \mp \frac{d}{2r}\cos\theta\right) \tag{12.3.5}$$

$$\frac{1}{\mathscr{r}_\pm} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \tag{12.3.6}$$

Then:

$$\cos\left(\omega t - \frac{\mathscr{r}_{\pm}}{c}\right) = \cos\left[\omega t - \frac{\omega r}{c}\left(1 \mp \frac{d}{2r}\cos\theta\right)\right] \tag{12.3.7}$$

$$= \cos\left[\omega t - \frac{\omega r}{c} \pm \frac{\omega d}{2c}\cos\theta\right)\right] \tag{12.3.8}$$

$$= \cos\left[\omega t - \frac{\omega r}{c}\right]\cos\left[\frac{\omega d}{2c}\cos\theta\right)\right] \mp \sin\left[\omega t - \frac{\omega r}{c}\right]\sin\left[\frac{\omega d}{2c}\cos\theta\right)\right] \tag{12.3.9}$$

We also expect that $d \ll \lambda = \frac{2\pi c}{\omega}$ implying that $d \ll \frac{c}{\omega}$. This allows us to write that:

$$\cos\left(\omega t - \frac{\mathscr{r}_{\pm}}{c}\right) \approx \cos\left[\omega t - \frac{\omega r}{c}\right] \mp \frac{\omega d}{2c}\cos\theta \sin\left[\omega t - \frac{\omega r}{c}\right] \tag{12.3.10}$$

Consequently the scalar potential of the oscillating dipole will become:

$$V(r,\theta,\phi) = \frac{q_0}{4\pi\varepsilon_0}\left\{\frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)\left(\cos\left[\omega t - \frac{\omega r}{c}\right] - \frac{\omega d}{2c}\cos\theta\sin\left[\omega t - \frac{\omega r}{c}\right]\right)\right. \tag{12.3.11}$$

$$\left. - \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)\left(\cos\left[\omega t - \frac{\omega r}{c}\right] + \frac{\omega d}{2c}\cos\theta\sin\left[\omega t - \frac{\omega r}{c}\right]\right)\right\} \tag{12.3.12}$$

which after some algebra gives:

$$V(r,\theta,\phi) = \frac{q_0\cos\theta}{4\pi\varepsilon_0}\left\{\frac{d}{r^2}\cos\left[\omega t - \frac{\omega r}{c}\right] - \frac{\omega d}{rc}\sin\left[\omega t - \frac{\omega r}{c}\right]\right\} \tag{12.3.13}$$

or more simply:

$$V(r,\theta,\phi) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left\{\frac{1}{r}\cos\left[\omega t - \frac{\omega r}{c}\right] - \frac{\omega}{c}\sin\left[\omega t - \frac{\omega r}{c}\right]\right\} \tag{12.3.14}$$

If we are interested in the far field approximation where $r \gg \lambda \sim \frac{c}{\omega}$ then the cosine term in (12.3.14) will vanish:

$$V(r,\theta,\phi) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\frac{\omega}{c}\left\{\cancelto{0}{\frac{c}{\omega r}}\cos\left[\omega t - \frac{\omega r}{c}\right] - \sin\left[\omega t - \frac{\omega r}{c}\right]\right\} \tag{12.3.15}$$

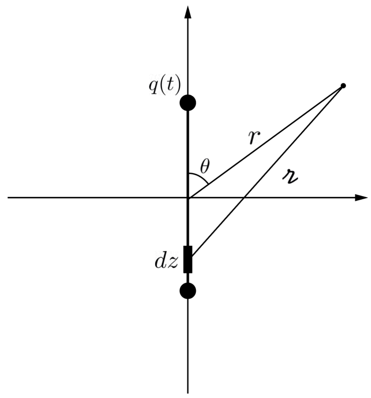
giving:

$$\boxed{V(r,\theta,\phi) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\frac{\cos\theta}{r}\sin\left[\omega t - \frac{\omega r}{c}\right]} \tag{12.3.16}$$

To find the vector potential, we must firstly find the current flowing through the wire:

$$\mathbf{I}(t) = \frac{dq}{dt}\hat{\mathbf{z}} = -q_0\omega \sin\omega t\hat{\mathbf{z}} \tag{12.3.17}$$

Hence:



$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega\sin(\omega(t-\mathscr{r}\,/c))}{\mathscr{r}}dz\hat{\mathbf{z}} \tag{12.3.18}$$

where we may approximate $z \ll r$ giving

$$\mathscr{r}\; = \sqrt{r^2 - rz\cos\theta + z^2} \approx r\left(1 - \frac{z}{2r}\cos\theta\right) \tag{12.3.19}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 - rz\cos\theta + z^2}} \approx \frac{1}{r}\left(1 + \frac{z}{2r}\cos\theta\right) \tag{12.3.20}$$

We may then write

$$\frac{\sin(\omega(t-\mathscr{r}\,/c))}{\mathscr{r}} \approx \frac{1}{r}\left(1 + \frac{z}{2r}\cos\theta\right)\sin\left[\omega t - \frac{\omega r}{c}\left(1 - \frac{z}{2r}\cos\theta\right)\right] \tag{12.3.21}$$

$$= \frac{1}{r}\left(1 + \frac{z}{2r}\cos\theta\right)\left(\sin\left[\omega t - \frac{\omega r}{c}\right]\cos\left[\frac{\omega z}{2c}\cos\theta\right]\right. \tag{12.3.22}$$

$$\left. + \cos\left[\omega t - \frac{\omega r}{c}\right]\sin\left[\frac{\omega z}{2c}\cos\theta\right]\right) \tag{12.3.23}$$

In the limit as $z \ll \frac{c}{\omega}$ one finds that:

$$\frac{\sin(\omega(t-\mathscr{r}\,/c))}{\mathscr{r}} \approx \frac{1}{r}\left(1 + \frac{z}{2r}\cos\theta\right)\left(\sin\left[\omega t - \frac{\omega r}{c}\right] + \cos\left[\omega t - \frac{\omega r}{c}\right]\frac{\omega z}{2c}\cos\theta\right) \tag{12.3.24}$$

We know that only terms of order d^2 will be kept, and since z will be integrated to d^2 we should only keep terms of z to order 0 (constants):

$$\frac{\sin(\omega(t-\mathscr{r}\,/c))}{\mathscr{r}} \approx \frac{1}{r}\sin\left[\omega t - \frac{\omega r}{c}\right] \tag{12.3.25}$$

Consequently:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{-q_0\omega}{r}\sin\left[\omega t - \frac{\omega r}{c}\right]\int_{-d/2}^{d/2} dz\hat{\mathbf{z}} \tag{12.3.26}$$

$$= \frac{\mu_0}{4\pi}\frac{-q_0\omega d}{r}\sin\left[\omega t - \frac{\omega r}{c}\right]\hat{\mathbf{z}} \tag{12.3.27}$$

giving:

$$\boxed{\mathbf{A} = \frac{\mu_0 p_0\omega}{4\pi r}\sin\left[\omega t - \frac{\omega r}{c}\right]\hat{\mathbf{z}}} \tag{12.3.28}$$

We have fully solved the problem now, it only remains to compute the fields from the potentials.

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} \tag{12.3.29}$$

$$= -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left(\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\sin\left[\omega t - \frac{\omega r}{c}\right]\right)\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega t - \frac{\omega r}{c}\right]\Bigg)\hat{\boldsymbol{\theta}} \tag{12.3.30}$$

$$= -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left[\cos\theta\left(-\frac{1}{r^2}\sin\left[\omega t - \frac{\omega r}{c}\right] - \frac{\omega}{rc}\cos\left[\omega t - \frac{\omega r}{c}\right]\right)\hat{\mathbf{r}}\right. \tag{12.3.31}$$

$$\left. - \frac{\sin\theta}{r^2}\sin\left[\omega t - \frac{\omega r}{c}\right]\hat{\boldsymbol{\theta}}\right] \tag{12.3.32}$$

Using the approximation $r \gg \frac{c}{\omega}$ then:

$$\nabla V \approx \frac{p_0\omega^2}{4\pi\varepsilon_0 c^2}\frac{\cos\theta}{r}\cos\left[\omega t - \frac{\omega r}{c}\right]\hat{\mathbf{r}} = \frac{\mu_0 p_0\omega^2}{4\pi}\frac{\cos\theta}{r}\cos\left[\omega t - \frac{\omega r}{c}\right]\hat{\mathbf{r}} \tag{12.3.33}$$

Similarly:

$$\nabla\times\mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{12.3.34}$$

$$= -\frac{\mu_0 p_0\omega}{4\pi r}\left[\frac{\omega}{c}\sin\theta\cos\left[\omega t - \frac{\omega r}{c}\right] + \frac{\sin\theta}{r}\sin\left[\omega t - \frac{\omega r}{c}\right]\right] \tag{12.3.35}$$

$$\approx -\frac{\mu_0 p_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega t - \frac{\omega r}{c}\right]\hat{\boldsymbol{\phi}} \tag{12.3.36}$$

Also:

$$\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos\left[\omega t - \frac{\omega r}{c}\right]\hat{\mathbf{z}}\underbrace{(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})}_{\hat{\mathbf{z}}} \tag{12.3.37}$$

Consequently using the Lorenz gauge:

$$\boxed{\mathbf{E} = -\frac{\mu_0 p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\boldsymbol{\theta}}} \tag{12.3.38}$$

and

$$\boxed{\mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\boldsymbol{\varphi}}} \tag{12.3.39}$$

The Poynting vector becomes:

$$\mathbf{S} = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\boldsymbol{\varphi}}\right\}^2\hat{\mathbf{r}} \tag{12.3.40}$$

and since $\langle\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\rangle = \frac{1}{2}$ over a cycle we find that:

$$\langle\mathbf{S}\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \tag{12.3.41}$$

The total power radiated is then given by the integral of $\langle\mathbf{S}\rangle$ over an arbitrary spherical surface

$$\langle P\rangle = \frac{\mu_0 p_0^2\omega^2}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta d\theta d\phi\hat{\mathbf{r}} = \frac{\mu_0 p_0^2\omega^4}{12\pi c} \tag{12.3.42}$$

where we used the integral

$$\int_0^\pi \sin^3\theta d\theta = \frac{4}{3} \tag{12.3.43}$$

12.4 Magnetic dipole radiation

Suppose we have a wire of radius b carrying a current:

$$I(t) = I_0\cos\omega t \tag{12.4.1}$$

giving a dipole moment:

$$\mathbf{m}(t) = \pi b^2 I_0\cos\omega t\hat{\mathbf{z}} \tag{12.4.2}$$

as shown below: The vector potential then reads:

$$\mathbf{A} = \frac{\mu_0 I_0}{4\pi}\int\frac{\cos(\omega t - \omega\mathscr{r}/c)}{\mathscr{r}}d\mathbf{l}' \tag{12.4.3}$$

Now

$$\mathscr{r} = \sqrt{r^2 - 2rb\cos\psi + b^2} \implies \frac{1}{\mathscr{r}} = \frac{1}{r}\frac{1}{\sqrt{1 - \frac{2b}{r}\cos\theta + \frac{b^2}{r^2}}} \tag{12.4.4}$$

Taking $b \ll r$ we may then approximate these distances as:

$$\mathscr{r} \approx r\left(1 - \frac{b}{r}\cos\psi\right) \tag{12.4.5}$$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r}\left(1 + \frac{b}{r}\cos\psi\right) \tag{12.4.6}$$

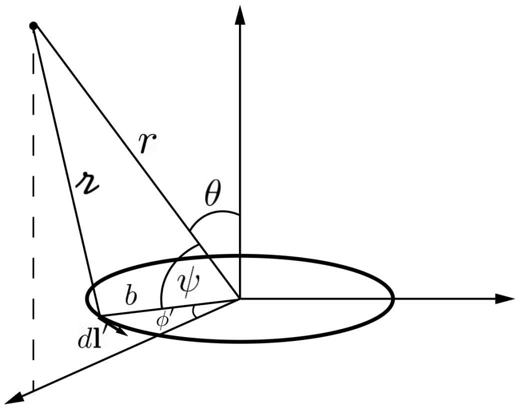


Figure 12.2. Magnetic dipole radiation

To find $\cos\psi$ we note that:

$$\mathbf{r}\cdot\mathbf{b} = rb\cos\psi \tag{12.4.7}$$

but also:

$$\mathbf{r}\cdot\mathbf{b} = (r\sin\theta\hat{\mathbf{x}} + r\cos\theta\hat{\mathbf{z}})\cdot(b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}) = rb\sin\theta\cos\phi' \tag{12.4.8}$$

implying that $\cos\psi = \sin\theta\cos\phi'$. Hence:

$$\mathscr{r} \approx r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right) \tag{12.4.9}$$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right) \tag{12.4.10}$$

Then:

$$\cos\left(\omega t - \frac{\omega\mathscr{r}}{c}\right) = \cos\left[\omega t - \frac{\omega r}{c}\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right)\right] \tag{12.4.11}$$

$$= \cos\left[\omega t - \frac{\omega r}{c} + \frac{\omega b}{c}\sin\theta\cos\phi'\right)\Bigg] \tag{12.4.12}$$

$$= \cos\left[\omega t - \frac{\omega r}{c}\right]\cos\left[\frac{\omega b}{c}\sin\theta\cos\phi'\right] - \sin\left[\omega t - \frac{\omega r}{c}\right]\sin\left[\frac{\omega b}{c}\sin\theta\cos\phi'\right] \tag{12.4.13}$$

We also expect that $b \ll \lambda = \frac{2\pi c}{\omega}$ implying that $b \ll \frac{c}{\omega}$. This allows us to write that:

$$\cos\left(\omega t - \frac{\omega\mathscr{r}}{c}\right) \approx \cos\left[\omega t - \frac{\omega r}{c}\right] - \frac{\omega b}{c}\sin\theta\cos\phi'\sin\left[\omega t - \frac{\omega r}{c}\right] \tag{12.4.14}$$

Consequently the scalar potential of the oscillating dipole will become:

$$\mathbf{A} = \frac{\mu_0 I_0}{4\pi}\int_0^{2\pi}\frac{1}{r}\left(1+\frac{b}{r}\sin\theta\cos\phi'\right)\left(\cos\left[\omega t-\frac{\omega r}{c}\right]-\frac{\omega b}{c}\sin\theta\cos\phi'\sin\left[\omega t-\frac{\omega r}{c}\right]\right)b\hat{\boldsymbol{\phi}}'d\phi' \tag{12.4.15}$$

Now due to the symmetry of the configuration we expect there to only be a component along $\hat{\mathbf{y}}$ for a point $\mathbf{r}$ above the x-axis. Consequently

$$\mathbf{A} = \frac{\mu_0 I_0 b}{4\pi r}\int_0^{2\pi}\left(1+\frac{b}{r}\sin\theta\cos\phi'\right)\left(\cos\left[\omega t-\frac{\omega r}{c}\right]-\frac{\omega b}{c}\sin\theta\cos\phi'\sin\left[\omega t-\frac{\omega r}{c}\right]\right)\cos\phi' d\phi'\hat{\mathbf{y}} \tag{12.4.16}$$

We may ignore the term

$$\int_0^{2\pi}\frac{b}{r}\frac{\omega b}{c}\sin^2\theta\cos^3\phi'\sin\left[\omega t-\frac{\omega r}{c}\right]d\phi' = 0 \tag{12.4.17}$$

which is of order b^2. Using the integrals:

$$\int_0^{2\pi}\cos\phi' d\phi' = 0,\ \int_0^{2\pi}\cos^2\phi' d\phi' = \pi \tag{12.4.18}$$

we find that:

$$\mathbf{A} = \frac{\mu_0 I_0 b}{4\pi r}\left\{-\pi\frac{\omega b}{c}\sin\theta\sin\left[\omega t-\frac{\omega r}{c}\right]+\frac{b}{r}\sin\theta\cos\left[\omega t-\frac{\omega r}{c}\right]\right\}\hat{\mathbf{y}} \tag{12.4.19}$$

$$= \frac{\mu_0 I_0 b^2}{4r}\frac{\omega}{c}\left\{-\sin\theta\sin\left[\omega t-\frac{\omega r}{c}\right]+\frac{c}{\omega r}\sin\theta\cos\left[\omega t-\frac{\omega r}{c}\right]\right\}\hat{\mathbf{y}} \tag{12.4.20}$$

Hence in the far field limit as $r \gg \frac{c}{\omega}$ we find that:

$$\mathbf{A} = -\frac{\mu_0 m_0 \omega}{4\pi c}\frac{\sin\theta}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\mathbf{y}} \tag{12.4.21}$$

More generally, since the point was assumed to be directly above the x-axis where $\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}}$ we may write more generally that:

$$\boxed{\mathbf{A} = -\frac{\mu_0 m_0 \omega}{4\pi c}\frac{\sin\theta}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\phi}}} \tag{12.4.22}$$

using the rotational symmetry of the configuration (we can always rotate the coordinate axes to put the point above the x-axis, thus justifying the choice $\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}}$).

Therefore:

$$\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 m_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\phi}} \tag{12.4.23}$$

and

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\boldsymbol{\theta}} \tag{12.4.24}$$

$$= -\frac{\mu_0 m_0 \omega}{4\pi c}\left\{\frac{1}{r\sin\theta}\frac{2\sin\theta\cos\theta}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\mathbf{r}} + \frac{\omega}{rc}\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\theta}}\right\} \tag{12.4.25}$$

$$= -\frac{\mu_0 m_0 \omega}{4\pi c}\frac{\omega}{c}\left\{\cancelto{0}{\frac{c}{\omega}\frac{2\cos\theta}{r^2}}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\mathbf{r}} + \frac{\sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\theta}}\right\} \tag{12.4.26}$$

$$\approx -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\theta}} \tag{12.4.27}$$

Hence we use the Lorenz gauge to find that:

$$\boxed{\mathbf{E} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\phi}}} \tag{12.4.28}$$

and

$$\boxed{\mathbf{B} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\theta}}} \tag{12.4.29}$$

Now the Poynting vector reads:

$$\mathbf{S} = \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}^2\hat{\mathbf{r}} \tag{12.4.30}$$

which when averaged over a cycle gives:

$$\langle \mathbf{S} \rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \tag{12.4.31}$$

The power radiated is then:

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} \tag{12.4.32}$$

12.5 Scattering of light in the atmosphere

Suppose we have a plane wave $\mathbf{E} = E_{0x}\cos(kz - \omega t)\hat{\mathbf{x}}$ incident on a particle at the origin. The particle is composed of a positive nucleus and a negative electron cloud, consequently a dipole moment $\mathbf{p}$ will be induced when under the influence of this electromagnetic field. For sufficiently weak $\mathbf{E}$ we will have that:

$$\mathbf{p} = \alpha\mathbf{E} \tag{12.5.1}$$

where α is the polarizability of the particle.

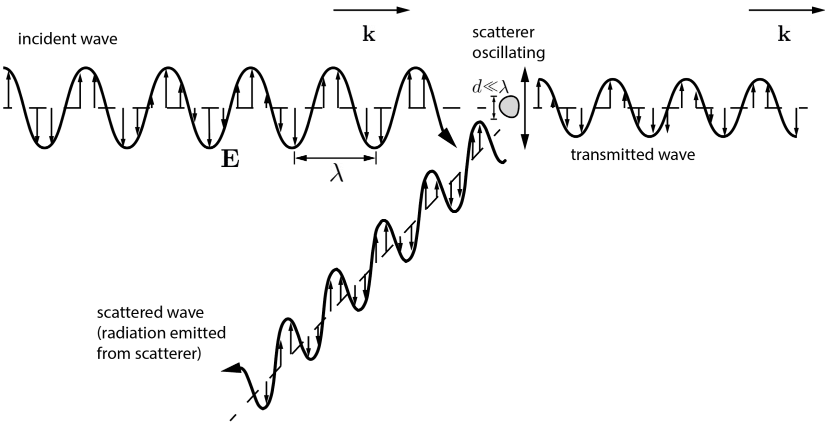


Figure 12.3. Process of rayleigh scattering for one particle.

Therefore, the particle will absorb some power from the impinging electromagnetic wave and behave as an oscillating dipole, with the same frequency ω, it will scatter light. The power scattered by the particle will then be:

$$\langle P\rangle = \frac{p_0^2\omega^4}{12\pi\varepsilon_0 c^3} = \frac{\alpha^2 E_{0x}^2\omega^4}{12\pi\varepsilon_0 c^3} \tag{12.5.2}$$

Since the incident power per unit area transported by the electromagnetic wave is:

$$\langle S\rangle = \frac{1}{2}\varepsilon_0 E_{0x}^2 c \tag{12.5.3}$$

then we may define the **scattering cross-section** σ as the area satisfying:

$$\langle P\rangle = \sigma \langle S\rangle \tag{12.5.4}$$

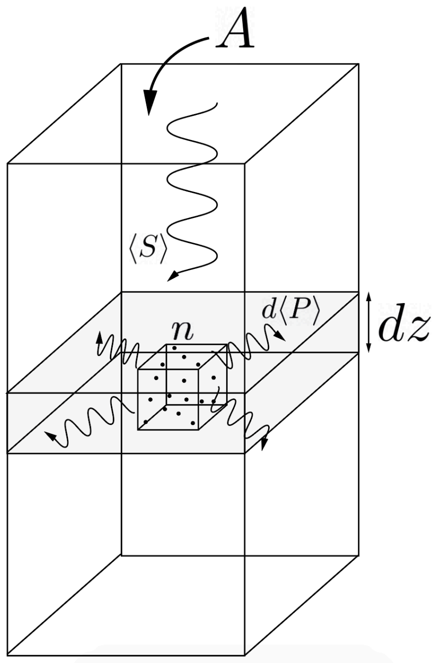
It is the ratio between the total scattered power and the incident power per unit area. In our present case it takes the value:

$$\sigma = \frac{\alpha^2 E_{0x}^2\omega^4}{12\pi\varepsilon_0 c^3}\frac{2}{\varepsilon_0 E_{0x}^2 c} = \frac{\alpha^2\omega^4}{6\pi\varepsilon_0^2 c^4} \tag{12.5.5}$$

Note that the scattered light will have the same frequency as the incident light, but simply a different direction of propagation.

Now consider a column of gas of cross section A illuminated from above, with particle density n. Then a tiny slab of height dz will have $nAdz$ scattering particles. Each of these will have a scattering cross-section σ so that:

$$d\langle P\rangle = (nAdz)\cdot\sigma\langle S\rangle \tag{12.5.6}$$



However, if $d\langle P\rangle$ is scattered then $d\langle S\rangle = -\frac{d\langle P\rangle}{A}$ and consequently:

$$d\langle S\rangle = -n\sigma\langle S\rangle\, dz \implies \langle S\rangle = \langle S\rangle_0\, e^{-z/L} \quad (12.5.7)$$

where $L = \frac{1}{n\sigma}$ is known as the scattering length, and $\langle S\rangle_0$ is the time averaged Poynting vector at the top of the column of gas. Hence the radiated power decreases exponentially.

This phenomenon known as Rayleigh scattering explains the color of the sky at varying times of the day.

When the sun is directly overhead (observer B), the incident sunlight will get scattered in the upper atmosphere where the particles are small enough for Rayleigh scattering to occur. Since the scattered power is frequency dependent with $\langle P\rangle \propto \langle\omega\rangle^4$, it follows that blue light with grater frequency will get scattered more than red light. Consequently when we look at the sky we will see the light scattered by the atmosphere which will mostly be blue in the visible spectrum. Looking at the sun directly however, we will look at the un-scattered light from the sun, which will be bright yellow.

When the sunlight is tangent to the earth (observer A), it will have to travel through a much longer portion of the atmosphere. Consequently a lot more of the blue light will get scattered by the particles in the atmosphere compared to when the sun is directly overhead. Looking at the sun directly and its surroundings we will then see the un-scattered light, which will be bright red. As we look farther away from the sun we will also start seeing part of the scattered light, which results in a bluish hue.

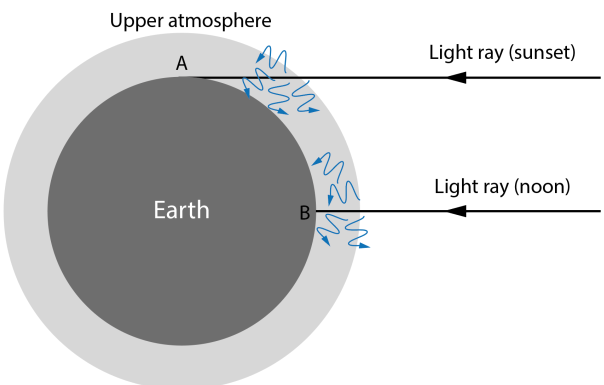


Figure 12.4. Rayleigh scattering in the upper atmosphere

12.6 Polarization of the sky

Sunlight is unpolarized, hence examining light beams travelling in some direction, the corresponding electric fields may oscillate in any direction in the plane perpendicular to the propagation direction. There are several possible such directions of linear polarization. We may resolve all these into two orthogonal directions $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$.

When a sunbeam hits a dust particle in the upper atmosphere, the particle will oscillate in the direction of polarization. Hence for an observer B perpendicular to the sunbeam, a dipole oscillating along $\hat{\mathbf{a}}_2$ due to the $\hat{\mathbf{a}}_2$ component of polarization will not be observed. This is because the electric field due to an oscillating dipole has a $\sin\theta$ term, which vanishes along the axis of oscillation. Hence the observer will only notice a polarization along $\hat{\mathbf{a}}_1$ (since $\mathbf{E} \parallel \hat{\boldsymbol{\theta}}$ for a radiation field), the light will be linearly polarized, and it will be polarized along the direction of oscillation, that is, perpendicular to the paper.

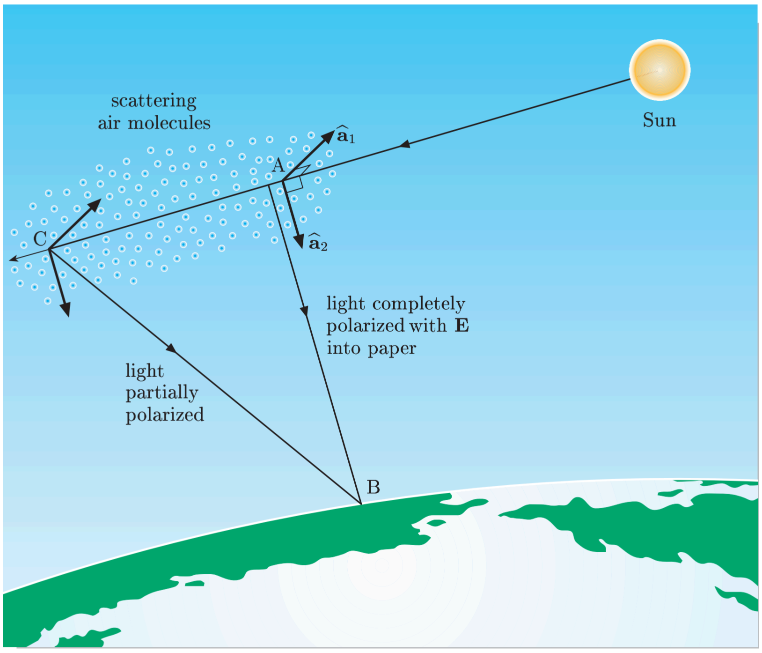


Figure 12.5. Polarization of the sky in the evening/morning

For an observer that is not perpendicular to the sunbeam, there will be a component of the electric field due to a dipole oscillating along $\hat{\mathbf{a}}_2$ and another component of the electric field due to a dipole oscillating along $\hat{\mathbf{a}}_1$. It will only be partially polarized.

The moral of the story is that observers perpendicular to a light beam that gets scattered will notice a linear polarization. In other words, light scattered at an angle of 90° from a light beam will be linearly polarized. Thus when looking at the sky in the morning and late evening (when the sun is at the east/west respectively), the sunbeams will be mostly horizontal, and consequently it will look polarized in the north-south direction ($\hat{\mathbf{a}}_1$).

Part IV

Wave optics

Interference

13

Scattering and Diffraction

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Covariant Electromagnetism

15

Acknowledgments

This is the most common positions for acknowledgments. A macro is available to maintain the same layout and spelling of the heading.

Note added. This is also a good position for notes added after the paper has been written.

Bibliography

[1] Author, *Title*, *J. Abbrev.* **vol** (year) pg.

[2] Author, *Title*, arxiv:1234.5678.

[3] Author, *Title*, Publisher (year).