Vol. 1 – CLASSICAL PHYSICS

The Undergraduate Companion to Theoretical Physics

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Contents

The enchanting charms of this sublime science reveal only to those who have the courage to go deeply into it.

— Carl Friedrich Gauss

References

Several textbooks, online courses/resources were referenced heavily (to the extend of making this text completely unoriginal, yet hopefully helpful for revision) throughout the writing of these lecture notes. Using a typical bibliography (research paper style) would be a formidable task.Pinpointing exactly where each reference as been used is quite difficult for such a large and well-referenced subject, and would probably change the writing style to a far too formal one for lecture notes. Therefore we instead list the most relevant below giving a brief comment on which topics they were mostly used for:

- L.D. Landau & E.M. Lifshitz *Course of Theoretical Physics Vol. 1*

  A short but crystal clear exposition of analytical mechanics.

Part I

Analytical mechanics

Lagrangian mechanics

1

1.1 Generalized coordinates

Consider a N-body classical system. We define $\{q_1, ..., q_n\}$ to be **generalised coordinates** of the system if the position vector for each particle is completely described by these coordinates (and explicitly time if necessary):

$$\mathbf{r}_i = \mathbf{r}_i(q_1, ..., q_n, t) \tag{1.1.1}$$

In 3D for example we should expect $6N$ such coordinates. Not all $6N$ of these coordinates however have to be independent of each other, varying just one of these could have an effect on the other coordinates too. This may for example be due to constraints on the system: an isolated system must conserve the position of the center of mass, thus fixing $3N$ coordinates by default. The number n of independent coordinates is known as the **degree of freedom** of the system. The space of coordinates $\mathbf{q} = (q_1, ..., q_n)$ is known as the **configuration space** or **phase space**. For example, a rigid body with three or more mass points has six degrees of freedom, three for the center of mass and three for the Euler angles. This choice however is not unique, we could have also chosen any other set of 6 independent coordinates. Note that this reduction was only possible because the system was rigid, the masses are all connected to each other, giving us $3(N-2)$ independent constraints eliminating $3(N-2)$ coordinates. Similarly, a compact disc rotating about a fixed axis has 3 degrees for the center of mass and 2 Euler angles.

In general, if one can reduce via constraints the number of generalised coordinates in $\mathbf{r}_i$ to the degrees of freedom then the system is **holonomic**. More rigorously, a constraint is said to be holonomic if it can be expressed as

$$f(q_i, t) = 0 \tag{1.1.2}$$

while it is non-holonomic otherwise. The constraint that a coin rolls without slipping for example is non-holonomic, so is the constraint that all particles in a confined gas remain inside a box, while requiring that the distance between a bob on the end of a rigid pendulum be fixed is holonomic. If t is eliminated in the process of imposing a holonomic constraint then the system is said to be **natural**, while if it introduces explicitly a dependence on time then it is **forced**.

To explore this further, let us differentiate (1.1.1)

$$\dot{\mathbf{r}}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \tag{1.1.3}$$

Here $\dot{q}_i$'s are the **generalised velocities** and together with the general coordinates they fully specify the configuration of the system at *any* future time. Note also that in the natural constraint case, differentiating with respect to generalised velocity gives

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{1.1.4}$$

which is a very useful identity! If the system is natural we also see that the kinetic energy takes the form

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_{i,j} \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \tag{1.1.5}$$

is quadratic and homogeneous in $\dot{q}_j$, while if the system is forced it will also include non-homogeneous terms.

1.2 Hamilton's principle and the Euler-Lagrange equations

Every system has a characteristic function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, known as a **Lagrangian**, from which we define the **action** functional

$$S[\mathbf{q}] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt,\ \mathbf{q}(t_1) = \mathbf{q}_1, \mathbf{q}(t_2) = \mathbf{q}_2 \tag{1.2.1}$$

for any given t_1, t_2. The constraint for the motion between t_1 and t_2 of the system is that it minimizes the action functional, this is the **Least action principle** or **Hamilton's principle**.

Note that the Lagrangian does not depend on higher order derivatives because the evolution of a system is fully determined by calculating the generalised coordinates and velocities.

The question of minimising an integral functional along the path between two points is a classic problem in the calculus of variations (see the Mathematical methods volume), and can be readily solved. We consider a small variation $\delta q_i(t)$ in the coordinate $q_i(t)$ subject to the boundary condition:

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \tag{1.2.2}$$

necessary to ensure that the perturbed path crosses the given end-points of motion. We

see that

$$\delta S[\mathbf{q}] = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \tag{1.2.3}$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt + \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt \tag{1.2.4}$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt \tag{1.2.5}$$

This must hold for arbitrary satisfying the boundary conditions, so the integrand must vanish

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{1.2.6}$$

These are the **Euler-Lagrange equations**, they are the equations of motion which minimise the action as required by Hamilton's principle.

1.3 The Lagrangian

Let's consider a single particle under the influence of a net force $\mathbf{F}$ from time t_0 to t_1. We are interested in the following integral

$$I[q] = \int_{t_0}^{t_1} T(q, \dot{q}, t)\, dt = \int_{t_0}^{t_1} \frac{1}{2} m \dot{\mathbf{r}}^2\, dt \tag{1.3.1}$$

and look at variations in the coordinate q_i, such that $\delta q_i(t_0) = \delta_i(t_1) = 0$ so that the end-points of the path are fixed. In other words we look at what happens to I when we slightly deform the path integrated over. Using the same arguments as in the previous section, we arrive at

$$\delta I[q] = \int_{t_0}^{t_1} \left[\frac{\partial T}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \right] \delta q_i dt \tag{1.3.2}$$

Moreover, note that the Work-Energy theorem requires

$$\delta I[q] = \int_{t_0}^{t_1} \frac{1}{2} m \dot{\mathbf{r}} \cdot \delta \dot{\mathbf{r}}\, dt = \int_{t_0}^{t} \delta W\, dt \tag{1.3.3}$$

so if we introduce the **generalised forces** F_i such that

$$\delta W = \sum_j F_j \delta q_j \tag{1.3.4}$$

then we finally find that

$$\int_{t_0}^{t_1} \left[\frac{\partial T}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \right] \delta q_i dt = \int_{t_0}^{t_1} F_i \delta q_i dt \tag{1.3.5}$$

This must hold for any variation $\delta q_i(t)$ satisfying the necessary boundary conditions, so both integrands must be equal to each other

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = F_i \tag{1.3.6}$$

Suppose that $\mathbf{F}$ can be split into a conservative component, $\mathbf{F}^c = -\nabla V$ [1], and a non-conservative component $\mathbf{F}^{nc}$. Then we see that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i^{nc} \tag{1.3.7}$$

where $L = T - V$. But this is exactly the Euler-Lagrange equations (generalised to non-conservative forces)! It follows that for the mechanical systems in consideration, the lagrangian is given by

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, \dot{q}, t) \tag{1.3.8}$$

Note that sometimes even the non-conservative potential can be written as

$$F_i^{nc} = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i} \tag{1.3.9}$$

in which case we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1.3.10}$$

This derivation is easily extended to N-body systems, where i now runs over the degrees of freedom rather than from 1 to 3.

Note that Lagrangians can differ by a total time derivative and still define the same physical system. Indeed, suppose that a system with n degrees of freedom has Lagrangian $L(q_i, \dot{q}_i, t)$. Define a new Lagrangian by

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d}{dt}F(q_i, t) \tag{1.3.11}$$

$$= L(q_i, \dot{q}_i, t) + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\dot{q}_i \tag{1.3.12}$$

Then we see that

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) \tag{1.3.13}$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) \tag{1.3.14}$$

implying that

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \iff \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) \tag{1.3.15}$$

[1] we assume that V is independent of $\dot{q}_i$ which is true for conservative forces, but not true in general. For example the Lorentz force is velocity-dependent and so will its potential be too.

as desired.

Moreover, we should also expect our equations of motion to be covariant under a change of coordinates. Indeed the minimisation of the action along a path is independent of what coordinates we use to describe the path, so even though the equation for the path will change, it should do so only covariantly (so that the same physical path is described). To check this, suppose we start with a Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$, and we perform a change of variables so that $\mathbf{q}' = \mathbf{q}'(\mathbf{q}, t) \iff \mathbf{q} = \mathbf{q}(\mathbf{q}', t)$. Using the chain rule we see that

$$\dot{q}_i = \frac{\partial q_i}{\partial q'_j}\dot{q}'_j + \frac{\partial q_i}{\partial t} \implies \frac{\partial \dot{q}_i}{\partial q'_j} = \frac{\partial q_i}{\partial q'_j} \tag{1.3.16}$$

We will also need the following identity

$$\frac{\partial \dot{q}_j}{\partial q'_i} = \frac{\partial}{\partial q'_i}\left(\frac{\partial q_j}{\partial q'_k}\dot{q}'_k\right) + \frac{\partial^2 q_j}{\partial q'_i \partial t} \tag{1.3.17}$$

$$= \dot{q}_k \frac{\partial}{\partial q'_k}\left(\frac{\partial q_j}{\partial q'_i}\right) + \frac{\partial}{\partial t}\left(\frac{\partial q_j}{\partial q'_i}\right) \tag{1.3.18}$$

$$= \frac{d}{dt}\left(\frac{\partial q_j}{\partial q'_i}\right) = \frac{d}{dt}\left(\frac{\partial \dot{q}_j}{\partial \dot{q}'_i}\right) \tag{1.3.19}$$

where we used (1.3.16) in going to the last step. Using the chain rule we also see that

$$\frac{\partial L'}{\partial q'_i} = \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial q'_i} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q'_i} \tag{1.3.20}$$

and similarly

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}'_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_i}{\partial q'_j}\right) \tag{1.3.21}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\frac{\partial \dot{q}_i}{\partial \dot{q}'_j} + \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\left(\frac{\partial \dot{q}_i}{\partial q'_j}\right) \tag{1.3.22}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\frac{\partial q_i}{\partial q'_j} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q'_i} \tag{1.3.23}$$

$$\tag{1.3.24}$$

Therefore the Euler-Lagrange equations for a Lagrangian transforms covariantly (as a scalar):

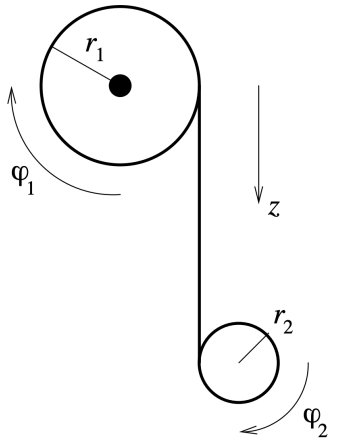
$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}'_i}\right) - \frac{\partial L'}{\partial q'_i} = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\frac{\partial q_j}{\partial q'_i} \tag{1.3.25}$$

1.4 Examples

The best way to learn how Lagrangian mechanics works is solving a lot of problems. Therefore, here we present some interesting examples of Lagrangians in action.

Example 1: double yo-yo

We consider two homogeneous cylinders of mass m and radii r_1, r_2 connected with a string. The first cylinder is fixed to a frictionless axis, while the lower one is allowed to fall so that the string it is connected to unravels as it descends.



The first step is to set-up the coordinate system. At first it may seem like we have three degrees of freedom, namely φ_1, φ_2 and z_2. However, since the string is inextensible we have the no-slip condition $\dot{z}_2 = r_1\dot{\varphi}_1 + r_2\dot{\varphi}_2$, which we integrate to give $z_2 = r_1\varphi_1 + r_2\varphi_2$. Thus we really only have two degrees of freedom, φ_1, φ_2.

The second step is to obtain the Lagrangian. We see that the kinetic energy is

$$T = \frac{1}{2}I_1\dot{\varphi}_1^2 + \frac{1}{2}I_2\dot{\varphi}_2^2 + \frac{1}{2}m(r_1\dot{\varphi}_1 + r_2\dot{\varphi}_2)^2 \tag{1.4.1}$$

while the potential energy is

$$V = -mg(r_1\phi_1 + r_2\phi_2) \tag{1.4.2}$$

We also have a non-conservative force, the tension in the string, Consequently the Lagrangian is

$$L(\varphi_i, \dot{\varphi}_i) = \frac{1}{2}I_1\dot{\varphi}_1^2 + \frac{1}{2}I_2\dot{\varphi}_2^2 + \frac{1}{2}m(r_1\dot{\varphi}_1 + r_2\dot{\varphi}_2)^2 + mg(r_1\varphi_1 + r_2\varphi_2) \tag{1.4.3}$$

The Euler-Lagrange equations then give the following equations of motion:

$$\frac{d}{dt}\Big((I_1 + mr_1^2)\dot{\varphi}_1 + mr_1r_2\dot{\varphi}_2\Big) = mgr_1 \tag{1.4.4}$$

$$\frac{d}{dt}\Big((I_2 + mr_2^2)\dot{\varphi}_2 + mr_1r_2\dot{\varphi}_1\Big) = mgr_2 \tag{1.4.5}$$

which we rewrite as a linear system

$$\begin{cases}(I_1 + mr_1^2)\ddot{\varphi}_1 + mr_1r_2\ddot{\varphi}_2 = mgr_1\\ (I_2 + mr_2^2)\ddot{\varphi}_2 + mr_1r_2\ddot{\varphi}_1 = mgr_2\end{cases} \implies \begin{pmatrix}I_1 + mr_1^2 & mr_1r_2\\ mr_1r_2 & I_2 + mr_2^2\end{pmatrix}\ddot{\boldsymbol{\varphi}} = mg\begin{pmatrix}r_1\\ r_2\end{pmatrix} \tag{1.4.6}$$

The solution is found to be

$$\ddot{\varphi}_1 = g\frac{mr_2 - \frac{I_2+mr_2^2}{r_2}}{mr_1r_2 - \frac{(I_1+mr_1^2)(I_2+mr_2^2)}{mr_1r_2}} \tag{1.4.7}$$

$$\ddot{\varphi}_2 = g\frac{mr_1 - \frac{I_1+mr_1^2}{r_1}}{mr_1r_2 - \frac{(I_1+mr_1^2)(I_2+mr_2^2)}{mr_1r_2}} \tag{1.4.8}$$

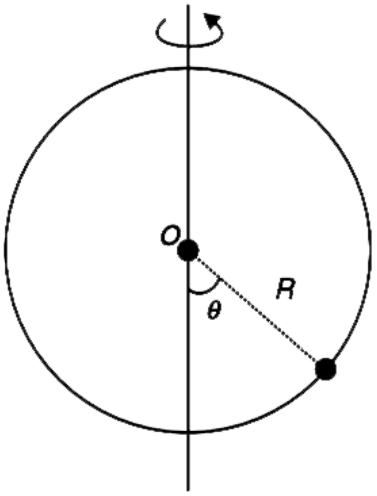
For a homogeneous cylinder, we have that $I_1 = \frac{1}{2}mr_1^2$ and $I_2 = \frac{1}{2}mr_2^2$, so that

$$r_1\ddot{\phi}_1 = r_2\ddot{\phi}_2 = \frac{2}{5}g \implies \ddot{z} = \frac{2}{5}g \tag{1.4.9}$$

Hence the lower cylinder will fall down with an acceleration of $\frac{2}{5}g$, which is less than the expected g, meaning that the tension force in the string must have been equal to $T = \frac{3}{5}mg$.

Example 2: particle sliding on rotating ring

A point particle of mass m is constrained to move on a frictionless circular wire of radius R spinning with constant angular speed ω about the vertical axis.



Firstly, the azimuthal coordinate ϕ of the mass is given by $\dot{\phi} = \omega \implies \phi = \omega t(\mod 2\pi)$. We expect θ to be the only remaining degree of freedom, the angle of the line between the mass and the center of the ring with the vertical axis. The kinetic energy is

$$T = \frac{1}{2}m(\omega R\sin\theta)^2 + \frac{1}{2}m(R\dot{\theta})^2 \tag{1.4.10}$$

while the potential energy is

$$V = -mgR\cos\theta \tag{1.4.11}$$

giving the following Lagrangian

$$L = \frac{1}{2}m\omega^2R^2\sin^2\theta + \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta \tag{1.4.12}$$

The Euler-Lagrange equation reads

$$mR^2\ddot{\theta} = m\omega^2 R^2 \cos\theta \sin\theta - mgR\sin\theta \tag{1.4.13}$$

which simplifies to

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta \tag{1.4.14}$$

Unfortunately this equation cannot be solved further. Nevertheless we can still find some interesting solutions, one of which are stationary solutions where $\theta(t) = \theta_0$, the mass is still relative to the ring it is sliding on. These are given by solving $\ddot{\theta} = 0$:

$$\left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta = 0 \tag{1.4.15}$$

One solution is when $\theta = 0$, which corresponds to the mass starting at the bottom of the ring. Similarly we also get $\theta = \pi$ which corresponds to the mass starting at the top of the ring. Due to absence of friction there is no force pushing it away, thus explaining these stationary solutions. Note however that any small perturbation will suddenly produce a net force which will change θ, thus hinting that these solutions may be unstable. The other solution occurs when $g \leq R\omega^2$ in which case

$$\cos\theta = \frac{g}{R\omega^2},\ \omega \geq \sqrt{\frac{g}{R}} \tag{1.4.16}$$

This corresponds to the bead starting at just the ring angle so that the its weight is perfectly balanced by the centripetal force produced by the rotational motion.

Let's now look at the sensitivities of these stationary values to small perturbations. For $\theta = 0$, let us define $\delta\theta$ to be a small perturbation in θ. It follows that $\ddot{\theta} = \ddot{\delta\theta}$ and $\sin(\delta\theta) \approx \delta\theta$ and $\cos(\delta\theta) \approx 1$. We then find that

$$\ddot{\delta\theta} = \left(\omega^2 - \frac{g}{R}\right)\delta\theta \tag{1.4.17}$$

Defining $\Omega = \sqrt{\omega^2 - \frac{g}{R}}$, if $\omega^2 > \frac{g}{R}$ we find that

$$\theta(t) = \delta\theta(t) = Ae^{\Omega t} + Be^{-\Omega t} \tag{1.4.18}$$

yielding unstable exponential solutions with characteristic life-time of $\tau \sim \frac{1}{\Omega}$. If instead $\omega^2 < \frac{g}{R}$ then

$$\theta(t) = \delta\theta(t) = Ae^{i\Omega t} + Be^{-i\Omega t} \tag{1.4.19}$$

so we get stable oscillations with frequency Ω. Similarly for $\theta = \pi$ we find that $\sin\delta\theta \approx -\delta\theta$ and $\cos\delta\theta = -1$ so that

$$\ddot{\delta\theta} = \left(\omega^2 + \frac{g}{R}\right)\delta\theta \tag{1.4.20}$$

so defining $\Delta = \sqrt{\omega^2 + \frac{g}{R}}$ we obtain another solution

$$\theta(t) = \pi + Ae^{\Delta t} + Be^{-\Delta t} \tag{1.4.21}$$

which is always unstable due to the positivity of the root in Δ.

For $\theta = \cos^{-1}\left(\frac{g}{R\omega^2}\right) \equiv \theta_0$, we let $\delta\theta$ be a small perturbation. Again $\ddot{\theta} = \delta\theta$, but this time we also have that

$$\sin(\theta_0 + \delta\theta) \approx \sin\theta_0 + \delta\theta\cos\theta_0,\ \cos(\theta_0 + \delta\theta) = \cos\theta_0 - \delta\theta\sin\theta_0 \tag{1.4.22}$$

implying that

$$\ddot{\delta\theta} = -\omega^2\delta\theta\sin\theta_0\left(\sin\theta_0 + \delta\theta\frac{g}{R\omega^2}\right) \tag{1.4.23}$$

$$\approx -\delta\theta\left(\omega^2 - \frac{g}{R}\right) \tag{1.4.24}$$

For $\omega^2 > \frac{g}{R}$, this is a simple harmonic oscillator with angular frequency $\Omega = \sqrt{\omega^2 - \frac{g}{R}}$:

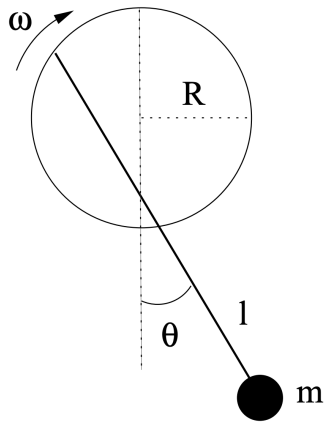
$$\theta(t) = \cos^{-1}\left(\frac{g}{R\omega^2}\right) + Ae^{i\Omega t} + Be^{-i\Omega t} \tag{1.4.25}$$

while if $\omega^2 < \frac{g}{R}$ we get an unstable solution

$$\theta(t) = \cos^{-1}\left(\frac{g}{R\omega^2}\right) + Ae^{\Omega t} + Be^{-\Omega t} \tag{1.4.26}$$

Example 3: pendulum attached to rotating disc

The pivot of a simple pendulum is attached to a disc of radius R, which rotates in the plane of the pendulum with angular velocity ω.



We start in cartesian coordinates, where the pivot has position $(R\sin\omega t, R\cos\omega t)$ and the mass has position (x, y) where

$$x = R\sin\omega t + l\sin\theta,\ y = R\cos\omega t - l\cos\theta \tag{1.4.27}$$

The kinetic energy then becomes

$$T = \frac{1}{2}m\left[(\omega R\cos\omega t + l\dot{\theta}\cos\theta)^2 + (\omega R\sin\omega t - l\dot{\theta}\sin\theta)^2\right] \tag{1.4.28}$$

$$= \frac{1}{2}m(\omega^2R^2 + \dot{\theta}^2l^2 + 2\omega\dot{\theta}Rl\cos(\theta+\omega t)) \tag{1.4.29}$$

while the potential energy is

$$V = mg(R\cos\omega t - l\cos\theta) \tag{1.4.30}$$

Therefore the Lagrangian can be written as

$$L = \frac{1}{2}m(\omega^2R^2 + \dot{\theta}^2l^2 + 2\omega\dot{\theta}Rl\cos(\theta+\omega t)) - mg(R\cos\omega t - l\cos\theta) \tag{1.4.31}$$

yielding the following equation of motion

$$l^2\ddot{\theta} - 2l\omega R(\dot{\theta}+\omega)\sin(\theta+\omega t) = -gl\sin\theta - 2l\omega R\dot{\theta}\sin(\theta+\omega t) \tag{1.4.32}$$

We can simplify it to

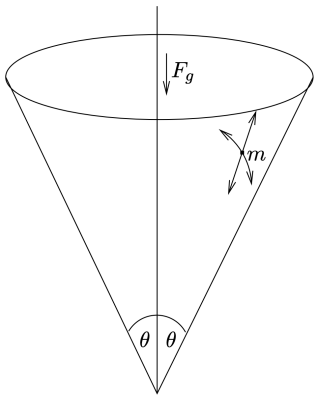
$$\ddot{\theta} - 2\omega^2\frac{R}{l}\sin(\theta+\omega t) + \frac{g}{l}\sin\theta = 0 \tag{1.4.33}$$

In the small angle limit, we can write $\sin\theta \approx \theta$ and $\sin(\theta+\omega t) \approx \sin\omega t + \theta\cos\omega t$ yielding

$$\ddot{\theta} - \left(2\omega^2\frac{R}{l}\cos\omega t - \frac{g}{l}\right)\theta = 2\omega^2\frac{R}{l}\sin\omega t \tag{1.4.34}$$

Example 4: particle gliding on cone

Consider a point particle of mass m gliding on the inside of a cone with aperture 2θ and friction coefficient μ.



Firstly, it is important to realise that the friction force on the cone is not conservative, and thus we will have to include it manually in the Euler-Lagrange equations. We adopt the coordinates r (distance from tip of the cone) and ϕ (azimuthal angle). Using the definition

of generalised forces we find

$$\delta W = F_r \delta r + F_\theta \delta\theta = -\mu mg \sin\theta(\delta r + r\sin\theta\delta\theta) \tag{1.4.35}$$

implying that $F_r = -\mu mg\sin\theta$ and $F_\theta = -\mu mgr\sin^2\theta$. The kinetic energy is

$$T = \frac{1}{2}m((\dot{\phi}r\sin\theta)^2 + \dot{r}^2) \tag{1.4.36}$$

while the potential energy is

$$V = mgr\cos\theta \tag{1.4.37}$$

Therefore, the Lagrangian can be written as

$$L = \frac{1}{2}m((r\dot{\phi}\sin\theta)^2 + \dot{r}^2) - mgr\cos\theta \tag{1.4.38}$$

The Euler-Lagrange equations are

$$\frac{d}{dt}(r^2\sin^2\theta\dot{\phi}) = -\mu gr\sin^2\theta \tag{1.4.39}$$

$$\ddot{r} - r\dot{\phi}^2\sin^2\theta + g\cos\theta = -\mu g\sin\theta \tag{1.4.40}$$

Firstly, note that $L_z = m\dot{\phi}r^2\sin^2\theta$ so the first equation implies that

$$\frac{dL_z}{dt} = -\mu gr\sin^2\theta \tag{1.4.41}$$

so if $\mu = 0$, then $L_z = l_z$ is a conserved quantity. In absence of friction the first equation simplifies to

$$mr^2\sin^2\theta\dot{\phi} = l_z \implies \dot{\phi} = \frac{l_z}{mr^2\sin^2\theta} \tag{1.4.42}$$

so that

$$\ddot{r} - \frac{l_z^2}{m^2r^3\sin^2\theta} + g\cos\theta = 0 \tag{1.4.43}$$

We seek solutions of constant radial distance r, and these are given by

$$\ddot{r} = 0 \implies r = \left(\frac{l_z^2}{m^2g\sin^2\theta\cos\theta}\right)^{1/3} \tag{1.4.44}$$

Example 5: sigma models

We consider a system with n degrees of freedom with a purely kinetic Lagrangian

$$L = \frac{1}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b \tag{1.4.45}$$

where $g_{ab}(\mathbf{q})$ is a coordinate-dependent symmetric metric in configuration space which we can assume to be invertible $g^{ab}g_{bc} = \delta^a_c$. The Euler-Lagrange equation for q^i reads

$$\frac{\partial L}{\partial q^i} = \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k,\ \frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j \implies g_{ij}\ddot{q}^j + \frac{\partial g_{ij}}{\partial q^k}\dot{q}^k\dot{q}^j = \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k \tag{1.4.46}$$

We now make use of the fact that $\frac{\partial g_{ij}}{\partial q^k}\dot{q}^k\dot{q}^j = \frac{\partial g_{ik}}{\partial q^j}\dot{q}^k\dot{q}^k$ to write

$$g_{ij}\ddot{q}^j + \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right)\dot{q}^j\dot{q}^k = 0 \tag{1.4.47}$$

Dotting to the left with g^{ai} we are left with the equation of motion

$$\ddot{q}^a + \frac{1}{2}g^{ai}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right)\dot{q}^j\dot{q}^k = 0 \tag{1.4.48}$$

The second term is a very familiar term from general relativity. Indeed we can define the Christoffel symbol

$$\Gamma^a_{jk} = \frac{1}{2}g^{ai}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right) \tag{1.4.49}$$

and write the equation of motion as a geodesic equation

$$\ddot{q}^a + \Gamma^a_{bc}\dot{q}^b\dot{q}^c = 0 \tag{1.4.50}$$

This is not surprising at all, since the geodesic gives the shortest path between two points, and the provided Lagrangian can be interpreted as the distance (squared) between two points in configuration space when integrated over time.

1.5 The Electromagnetic Lagrangian

Suppose a particle of charge e (to avoid confusion with the generalised coordinates) moves in a region of electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. Then it is a well known fact that the force on this charge will be given by the Lorentz force law

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{1.5.1}$$

In what follows we will work in the Coulomb gauge with potentials φ and $\mathbf{A}$ satisfying

$$\mathbf{E} = \nabla\varphi,\ \mathbf{B} = \nabla\times\mathbf{A} \tag{1.5.2}$$

The standard Lagrangian which yields the correct equation of motion is then

$$L = \frac{1}{2}m\mathbf{v}^2 + e(\varphi(\mathbf{r},t) + \mathbf{v}\cdot\mathbf{A}(\mathbf{r},t)) \tag{1.5.3}$$

To see why, we employ index notation and write (1.5.3) as

$$L = \frac{1}{2}m\dot{q}_i\dot{q}^i + e(\varphi + \dot{q}_iA^i) \tag{1.5.4}$$

It then follows that

$$\frac{\partial L}{\partial q_i} = e\left(\frac{\partial \varphi}{\partial q_i} + \dot{q}_j \frac{\partial A_j}{\partial q_i}\right) \tag{1.5.5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m\ddot{q}^i + e\dot{A}^i \tag{1.5.6}$$

and using some identities from tensor algebra we also find that

$$(\mathbf{v}\times\mathbf{B})^i = \epsilon^{ijk}\dot{q}_j(\epsilon_{klm}\partial^l A^m) = (\delta^i_l\delta^j_m - \delta^i_m\delta^j_l)\dot{q}_j\partial^l A^m) \tag{1.5.7}$$

$$= \dot{q}_j\partial^i A^j - \dot{q}_j\partial^j A^i = \dot{q}_j\partial^i A^j - \dot{A}^i \tag{1.5.8}$$

Consequently we find that

$$\frac{\partial L}{\partial q_i} = e(E_i + (\mathbf{v}\times\mathbf{B})^i + \dot{A}^i) \tag{1.5.9}$$

and therefore

$$m\ddot{q}^i = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})^i \tag{1.5.10}$$

1.6 Symmetries and Noether's theorem

We define a **constant of motion/conserved quantity** $J(q_i, \dot{q}_i, t)$ to be a quantity whose total time derivative vanishes

$$\frac{dJ}{dt} = \left(\frac{\partial J}{\partial q_j}\dot{q}_j + \frac{\partial J}{\partial \dot{q}_j}\ddot{q}_j\right) + \frac{\partial J}{\partial t} = 0 \tag{1.6.1}$$

for all q_i solving the Euler-Lagrange equations. This condition is known as an **on-shell condition**, it requires the $q_i, \dot{q}_i$ to lie on the shell of solutions to the equations of motion. This definition does not necessarily mean that a conserved quantity is time-independent, but rather that as we move along a path in phase space which solves to the equations of motion there is no variation in this quantity.

Importantly, one can relate conserved quantities of a Lagrangian to a special set of its symmetries. To define what we mean by a symmetry, let us consider a one-parameter family of maps

$$\varphi(\lambda) : q_i(t) \to Q_i(\lambda, t),\ m \in \mathbb{R} \tag{1.6.2}$$

with $Q_i(0,t) = q_i(t)$. For an infinitesimal symmetry transformation we may define the **symmetry variation** $\delta q_i(t)$ such that

$$q_i(t) \mapsto Q_i(t) = q_i(t) + \delta q_i(t) \tag{1.6.3}$$

We then say that φ is a **continuous symmetry** of a Lagrangian L if for infinitesimal variations

$$\delta L = L(Q_i(\lambda,t), \dot{q}_i(\lambda,t), t) - L(Q_i(\lambda,t), \dot{q}_i(\lambda,t), t) = \frac{d\Lambda}{dt} \tag{1.6.4}$$

so that the overall action is invariant up to a constant

$$\mathcal{S} = \int_{t_a}^{t_b} dt\ L(q_i, \dot{q}_i, t)\ dt = \int_{t_a}^{t_b} dt\ L(Q_i, \dot{Q}_i, t)\ dt + \Lambda|_{t_a}^{t_b} \tag{1.6.5}$$

Noether's theorem states that:

Noether's Theorem: every continuous symmetry of a Lagrangian L gives rise to a conserved constant of motion J.

Proof. Since φ is a continuous symmetry we can consider an infinitesimal transformation

$$q_i(t) \mapsto q_i(t) + \delta q_i(t) \tag{1.6.6}$$

We need the change δL of the Lagrangian under this map to be at most a total time derivative, so that the action will only change by a boundary term. Thus

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i(t) \tag{1.6.7}$$

$$= \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i(t) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i(t)\right) = \frac{d\Lambda}{dt} \tag{1.6.8}$$

Since this applies to $(q_i(t), \dot{q}_i(t))$ on-shell, we must require that the Euler-Lagrange equations are satisfied

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{1.6.9}$$

so substituting this into (1.6.8) we get that the first term vanishes, and thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i(t) - \Lambda\right) = 0 \tag{1.6.10}$$

We have found that the following quantity

$$J = \frac{\partial L}{\partial \dot{q}_i}\delta q_i(t) - \Lambda, \quad \text{where } \frac{d\Lambda}{dt} = \delta L \tag{1.6.11}$$

is a constant of motion, as required by Noether's theorem. □

Note that many textbooks actually use the stronger condition of requiring the Lagrangian to be **invariant** under the symmetry, which usually occurs for systems with spatial translational and rotational symmetry. In this case we have that $\delta L = 0$ and therefore

$$J = \frac{\partial L}{\partial \dot{q}_i}\delta q_i(t) \tag{1.6.12}$$

There are a wide array of commonly found symmetries which lead to the typical conservation laws one encounters in Newtonian mechanics. These will be explored below.

Time translational symmetry

We will consider the following map

$$\varphi(\lambda) : q_i(t) \mapsto q_i(t+\lambda) \tag{1.6.13}$$

and assume that this is a continuous symmetry of the Lagrangian, which is definitely the case when it does not explicitly depend on t. Assuming this, the infinitesimal variation of this symmetry is

$$\delta q_i(t) = \epsilon \dot{q}_i(t) \implies \delta \dot{q}_i(t) = \epsilon \ddot{q}_i(t) \tag{1.6.14}$$

and since the Lagrangian only depends on $q_i, \dot{q}_i$ we also find that

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \epsilon \frac{dL}{dt} = \frac{d\Lambda}{dt} \implies \Lambda = \epsilon L \tag{1.6.15}$$

Finally, using Noether's theorem we find that

$$J = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = p_i \dot{q}_i - L \tag{1.6.16}$$

is a conserved quantity, and is known as the Hamiltonian of the system correspondonging to its total energy. Systems that conserve energy are known as **conservative**.

Space translational symmetry

We now consider space translations along the ith direction, defined by the following map

$$\varphi(\lambda) : q_i(t) \mapsto q_i(t) + \lambda \tag{1.6.17}$$

and assume that they are symmetries of a Lagrangian independent of the coordinate q_i. The infinitesimal symmetry variations are

$$\delta q_i(t) = \epsilon \implies \delta L = \epsilon \frac{\partial L}{\partial q_i} = 0 \tag{1.6.18}$$

Noether's theorem then gives us the conserved quantity

$$J = \frac{\partial L}{\partial \dot{q}_i} = p_i \tag{1.6.19}$$

so the momentum conjugate to the direction of translation will be conserved. Intuitively, this follows from Newton's second law. If the Lagrangian is independent of a coordinate q_i then the corresponding generalised force will also vanish, so

$$\frac{\partial L}{\partial q_i} \equiv F_i = 0 \implies \frac{dp_i}{dt} = 0 \tag{1.6.20}$$

by Newton's second law, or equivalent the Euler-Lagrange equations $\dot{p}_i = F_i$.

Rotational symmetry

We now consider rotations about the axis **n** defined by the following map

$$\varphi(\lambda) : \mathbf{q}(t) \mapsto \mathbf{q}(t) + \lambda \mathbf{q}(t) \times \mathbf{n} \tag{1.6.21}$$

and assume that they are symmetries of a Lagrangian. In an isotropic system one can rotate the coordinate system without affecting the Lagrangian. This can occur if for example the Lagrangian only depends on $\mathbf{q}^2, \mathbf{q}_i \cdot \mathbf{q}_j$ etc...The infinitesimal variations are given by

$$\delta \mathbf{q}(t) = \theta \mathbf{q}(t) \times \mathbf{n} \implies \delta q_i(t) = \theta \epsilon_{ijk} q_j(t) n_k \tag{1.6.22}$$

Consequently we get that the Lagrangian variation is given by

$$\delta L = \theta \epsilon_{ijk} \left(\frac{\partial L}{\partial q_i} q_j(t) + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_j(t) \right) n_k = \frac{d}{dt} \left(\theta \epsilon_{ijk} \frac{\partial L}{\partial \dot{q}_i} q_j(t) n_k \right) \tag{1.6.23}$$

Requiring the Lagrangian to be invariant under rotations yields the following conserved quantity

$$J = -\epsilon_{ijk} p_i q_j n_k = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{p}) \tag{1.6.24}$$

which is just the component of angular momentum along **n**. If space is isotropic then this will hold for any **n** so angular momentum will be conserved.

1.7 Small oscillations, normal modes and stability

Once we find the equations of motion of a system using the Lagrangian, unless they are analytically solvable the most interesting thing to look at is the existence of equilibria i.e. solutions $\mathbf{q}(t) = \mathbf{q}_0$ which are constant in time. Determining the stability of these equilibria is yet another important question: if the system is slightly perturbed from this equilibrium point, does it return to its initial configuration or is it driven away from it? It turns out that to answer this question we must look at the dynamics of small angle oscillations.

Let's consider a natural, holonomic conservative system with n degrees of freedom described by the Lagrangian

$$L = \frac{1}{2} T_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j - V(q) \tag{1.7.1}$$

where T_{ij} is taken to be symmetric without loss of generality (positions commute). We define $\{\mathbf{q}\}$ to be **normal coordinates** of the system if T_{ij} is diagonal in these coordinates, and since symmetric matrices are always diagonalisable this can always be done by moving to the eigenbasis of T_{ij}. By rescaling the normal coordinates one can then always write the kinetic energy as

$$T = \frac{1}{2} \dot{q}_i \dot{q}^i \tag{1.7.2}$$

In the way we have written the Lagrangian, note that there can be terms in the potential $V(q)$ which actually arise from the kinetic energy, but are nevertheless independent of $\dot{q}_i$ e.g. a centrifugal barrier. The Euler-Lagrange equations are

$$\frac{d}{dt}(T_{ij}(\mathbf{q}) \dot{q}_j) - \frac{1}{2} \frac{\partial T_{kj}}{\partial q_i} \dot{q}_k \dot{q}_j = -\frac{\partial V}{\partial q_i} \tag{1.7.3}$$

For $\mathbf{q}_0$ to be an equilibrium point we require all time derivatives of $\mathbf{q}$ to vanish so that

$$\left.\frac{\partial V}{\partial q_i}\right|_{\mathbf{q}_0} = 0 \tag{1.7.4}$$

We can always shift our reference frame so that $\mathbf{q}_0 = 0$, in which case a Taylor expansion of the Lagrangian would yield

$$L \approx \frac{1}{2}T_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\mathbf{q}=0} q_i q_j + o(q^3) \tag{1.7.5}$$

where we ignored the potential at the equilibrium point $V(0)$, since the Lagrangian is defined up to additive constants. We have therefore derived an effective quadratic lagrangian near the equilibrium point which we can express as

$$L_{\text{eff}} = \frac{1}{2}\mathcal{T}_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\mathcal{V}_{ij}q_i q_j,\ \mathcal{T}_{ij} = T_{ij}(\mathbf{q}=0),\ \mathcal{V}_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\mathbf{q}=0} \tag{1.7.6}$$

or in matrix notation

$$\boxed{L_{\text{eff}} = \frac{1}{2}\dot{\mathbf{q}}^T\mathcal{T}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^T\mathcal{V}\mathbf{q}} \tag{1.7.7}$$

The equation of motion now become

$$\mathcal{T}_{ij}\ddot{q}_j = -\mathcal{V}_{ij}q_j \iff \ddot{\mathbf{q}} = -\mathcal{T}^{-1}\mathcal{V}\mathbf{q} \tag{1.7.8}$$

which is typical of a system of coupled oscillators. We therefore introduce the oscillator ansatz $\mathbf{q} = \mathbf{A}e^{i\omega t}$ consisting of all coordinates oscillating at the same frequency ω with amplitudes A_i. Such solutions are known as **normal modes** of the system, and any general solution of the system can be written as a superposition all these modes. Substituting this ansatz we see that

$$-\omega^2\mathbf{A} = -\mathcal{T}^{-1}\mathcal{V}\mathbf{A} \implies (\omega^2\mathcal{T} - \mathcal{V})\mathbf{A} = \mathbf{0} \tag{1.7.9}$$

The non-trivial solutions $\mathbf{A} \neq \mathbf{0}$ can be found by solving the characteristic equation

$$\det\left(\omega^2\mathcal{T} - \mathcal{V}\right) = 0 \tag{1.7.10}$$

and the corresponding eigenvectors $\mathbf{A}$ can then be found by solving

$$(\omega^2\mathcal{T} - \mathcal{V})\mathbf{A} = 0 \tag{1.7.11}$$

Since both $\mathcal{T}$ and $\mathcal{V}$ are symmetric, they are always diagonalisable (by the spectral theorem of linear algebra). Next note that $\mathcal{T}^{-1}\mathcal{V}$ is similar to $\mathcal{T}^{-1/2}\mathcal{V}\mathcal{T}^{-1/2}$, which is symmetric, and thus also diagonalisable with n real eigenvalues and n linearly independent eigenvectors. If $\omega^2 > 0$ then we obtain an oscillating mode, if $\omega^2 = 0$ we get a linear solution, while if $\omega^2 < 0$ we obtain an exponentially growing solution. It follows that if any of the eigenvalues of $\mathcal{T}^{-1}\mathcal{V}$ are negative then the equilibrium point has a linear instability in the direction of the corresponding eigenvector, while if all eigenvalues are positive then the equilibrium point is stable.

Hamiltonian mechanics

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The Canonical equations

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Central forces and scattering

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Rotational kinematics and non-inertial frames

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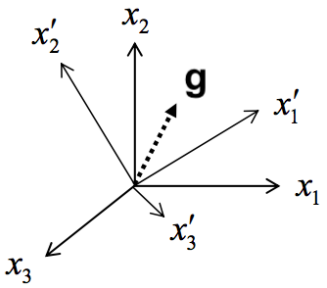
5.1 Orthogonal transformations

A three dimensional rigid body has 6 degrees of freedom, these could be for example 3 degrees from the center of mass coordinates and another 3 degrees specifying the orientation of the body relative to some coordinate axes. Therefore, to specify a rigid body's orientation it suffices to fix a set of axes to the rigid body, which rotates with it, and find the angles that it makes with a set of axes that are fixed in the laboratory frame.



For example, let Σ be the lab coordinate system with coordinates (x_1, x_2, x_3) and corresponding unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Similarly let Σ' be the coordinate system rotating with the rigid body, with coordinates (x_1', x_2', x_3') and corresponding unit vectors $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$. Given any vector $\mathbf{g}$, it can be expressed as

$$\mathbf{g} = g_i \mathbf{e}_i = g_j' \mathbf{e}_j \tag{5.1.1}$$

To relate the coordinates in the Σ and Σ' frames, note that

$$g_j' = \mathbf{g} \cdot \mathbf{e}_j' = g_i \mathbf{e}_i \cdot \mathbf{e}_j' = R_{ji} g_i, \text{ where } R_{ji} = \mathbf{e}_j' \cdot \mathbf{e}_i \tag{5.1.2}$$

Here R_{ij} are known as the **direction of cosines**, and form a matrix R. However, R_{ij} is a matrix with 9 entries in total, but we know that only three of these should be independent of each other. We solve this conundrum by noting that $\mathbf{g}^2$ should take the same value in both Σ and Σ':

$$\mathbf{g} \cdot \mathbf{g} = g_i' g_i' = (R_{ij} g_j)(R_{ik} g_k) = R_{ij} R_{ik} g_j g_k = \delta_{jk} g_j g_k \tag{5.1.3}$$

implying that

$$R_{ij} R_{ik} = \delta_{jk} \iff \mathsf{R}^T \mathsf{R} = \mathbb{1} \implies \det \mathsf{R} = \pm 1 \tag{5.1.4}$$

So the matrix R is orthogonal. It is a simple exercise to check that the set of orthogonal 3×3 matrices form a group under matrix multiplication, known as the **orthogonal group**

$O(3)$. Using the fact that

$$R_{ji} = \mathbf{e}'_j \cdot \mathbf{e}_i \implies \mathbf{e}_i = R_{ji}\mathbf{e}'_j \text{ and } \mathbf{e}'_j = R_{ji}\mathbf{e}_i \tag{5.1.5}$$

we get that

$$\mathbf{e}'_i \cdot \mathbf{e}'_j = (R_{ik}\mathbf{e}_k) \cdot (R_{jl}\mathbf{e}_l) = R_{ik}R_{jl}\delta_{kl} = R_{ik}R_{jk} = \delta_{ij} \tag{5.1.6}$$

so we see that orthogonal transformations maintain orthogonality.

Up until now we have been looking at the vector $\mathbf{g}$ as fixed in space, and the coordinate frame as rotating with the rigid body. This is known as the **passive point of view**, and corresponds to what the laboratory frame observer would see. Alternatively, one could think of the coordinate axes as being fixed, and the vector as rotating. This is known as the **active point of view**, and corresponds to what the observer on the rigid body would see. These two points of views are perfectly equivalent, but are useful in different scenarios.

In the active point of view, we have that $\mathbf{g} = g_i\mathbf{e}_i$, and since the $\mathbf{e}_i$ are fixed, we need to transform $\mathbf{g} \mapsto \mathbf{g}' = g'_i\mathbf{e}_i = R_{ij}g_j\mathbf{e}_i$ where

$$g'_i = R_{ij}g_j \iff \mathbf{g}' = \mathsf{R}\mathbf{g} \tag{5.1.7}$$

Note that in the passive picture, the unit vectors transformed contravariantly as $\mathbf{e}'_i = R_{ji}\mathbf{e}_j$, while in the active picture the coordinates transformed covariantly as $g'_i = R_{ij}g_j$. The fact that the two quantities transform oppositely is simple to understand, if we require the axes to be fixed then the transformation of the coordinates must be the inverse of the transformation of the axes required for the coordinates to be fixed.

Returning back to (5.1.4), note that an orthogonal matrix has a determinant of either $+1$ or -1. A typical orthogonal matrix with negative unit determinant is

$$\mathsf{I} = -\mathbb{1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.1.8}$$

which symmetric and thus idempotent (due to its orthogonality). Since $\mathsf{I}^{-1} = \mathsf{I}$, it follows that any matrix A can be written as $\mathsf{A} = \mathsf{I}\mathsf{B}$ where $\mathsf{B} = -\mathsf{A}$. Now suppose that A is an orthogonal matrix with $\det \mathsf{A} = +1$. Then it follows that if $\mathsf{B} = -\mathsf{A}$ then $\det \mathsf{B} = -1$. Similarly, if $\det \mathsf{B} = -1$ then since $\mathsf{B} = \mathsf{I}\mathsf{A}$ we get that $\det \mathsf{A} = 1$. Therefore, there is a one-to-one correspondence between positive and negative determinant orthogonal matrices. Since I inverts the coordinates, this means that negative unit determinant transformations are compositions of $\det\{\mathsf{R}\} = +1$ transformations and inversions, and are thus named improper transformations since they convert left handed bases to right handed ones and vice-versa. On the other hand, positive unit determinant transformations are called proper, or **rotations**, because they maintain the parity of the coordinate axes. The latter form a subgroup of the orthogonal group $O(3)$, known as the **special orthogonal group** $SO(3)$. This cannot be said for improper rotations, since for example $\mathsf{I}\mathsf{I} = \mathbb{1}$ which is not improper, violating the closure axiom of subgroups.

To understand why proper orthogonal transformations are the rotations we are so used to, let's consider two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. A rotation $\mathcal{R}$ can defined as a transformation

such that

$$\mathcal{R}(\mathbf{a}) \cdot \mathcal{R}(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \tag{5.1.9}$$
$$\mathcal{R}(\mathbf{a}) \times \mathcal{R}(\mathbf{b}) = \mathcal{R}(\mathbf{a} \times \mathbf{b}) \tag{5.1.10}$$

Therefore let $\mathsf{R} \in \mathrm{SO}(3)$ be a proper orthogonal transformation, and define

$$\mathbf{a}' = \mathsf{R}\mathbf{a},\ \mathrm{b}' = \mathsf{R}\mathbf{b} \tag{5.1.11}$$

It follows that

$$\mathbf{a}' \cdot \mathbf{b}' = a_i' b_i' = R_{ij} R_{ik} a_j b_k = \delta_{jk} a_j b_k = a_j b_j = \mathbf{a} \cdot \mathbf{b} \tag{5.1.12}$$

Similarly

$$(\mathbf{a}' \times \mathbf{b}')_i = \epsilon_{ijk} a_j' b_k' = \epsilon_{ijk} R_{jl} R_{km} a_l b_m \tag{5.1.13}$$

Now note that

$$\epsilon_{ijk} R_{il} R_{jm} R_{kn} = (\det \mathsf{R}) \epsilon_{lmn} = \epsilon_{lmn} \tag{5.1.14}$$

which implies

$$\epsilon_{ijk} R_{rl} R_{il} R_{jm} R_{kn} = \epsilon_{rjk} R_{jm} R_{kn} = R_{rl} \epsilon_{lmn} \tag{5.1.15}$$

Substituting this into (5.1.13) we find that

$$(\mathbf{a}' \times \mathbf{b}')_i = \epsilon_{ijk} R_{jl} R_{km} a_l b_m = R_{ir} \epsilon_{rlm} a_l b_m = (\mathsf{R}(\mathbf{a} \times \mathbf{b}))_i \tag{5.1.16}$$

as desired.

It turns out that rotations are the most general displacement that a rigid body can have which fixes only one point. This is known as Euler's theorem:

Euler's Theorem: the most general transformation of a rigid body which leaves one point fixed is a proper orthogonal transformation about an axis through the fixed point.

Suppose the rigid body starts out in some configuration, is transformed using a rotation R to some final configuration. We need to show that there will always be some point N with position vector $\mathbf{n}$ that is fixed by $\mathcal{R}$. This amounts to proving that there exists some $\mathbf{n}$ such that

$$\mathsf{R}\mathbf{n} = \mathbf{n} \implies \det(\mathsf{R} - \mathbb{1}) = 0 \tag{5.1.17}$$

or in other words, R is always diagonalisable. We show this by noting that

$$\mathsf{R} - \mathbb{1} = \mathsf{R}(\mathbb{1} - \mathsf{R}^T) = -\mathsf{R}(\mathsf{R} - \mathbb{1})^T \tag{5.1.18}$$
$$\implies \det(\mathsf{R} - \mathbb{1}) = -\det \mathsf{R} \det(\mathsf{R} - \mathbb{1}) = -\det(\mathsf{R} - \mathbb{1}) \tag{5.1.19}$$

and therefore $\det(\mathsf{R} - \mathbb{1}) = 0$.

5.2 Euler angles

We know that to specify the orientation of a rigid body, one needs to fix a coordinate frame to the rigid body and find the angle that its axes make with a laboratory coordinate frame. The **Euler angles** are a special set of angles that are practically very useful in defining the orientation of a rigid-body. Consider a set of axes $\mathbf{e}_a$. Then, any fixed origin rotation of these axes can be parametrised using Euler Angles. This parametrisation consists of a rotation around each axis:

$$\mathbf{e}_a \xrightarrow{R(\phi)} \mathbf{e}'_a \xrightarrow{R(\theta)} \mathbf{e}''_a \xrightarrow{R(\psi)} \tilde{\mathbf{e}}_a \tag{5.2.1}$$

Firstly, we rotate the axes by ϕ about $\mathbf{e}_3$, so that $\mathbf{e}_a = R(\phi)_{ab}\mathbf{e}_b$ where

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.2}$$

Secondly, we rotate the new axes by θ about $\mathbf{e}'_1$ so that $\mathbf{e}''_a = R(\theta)_{ab}\mathbf{e}'_b$ where:

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{5.2.3}$$

Finally, we rotate the newest axes by ψ about $\mathbf{e}''_3$ so that $\tilde{\mathbf{e}}_a = R(\psi)_{ab}\mathbf{e}''_b$,where:

$$R(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.4}$$

Visually this sequence of rotations amounts to

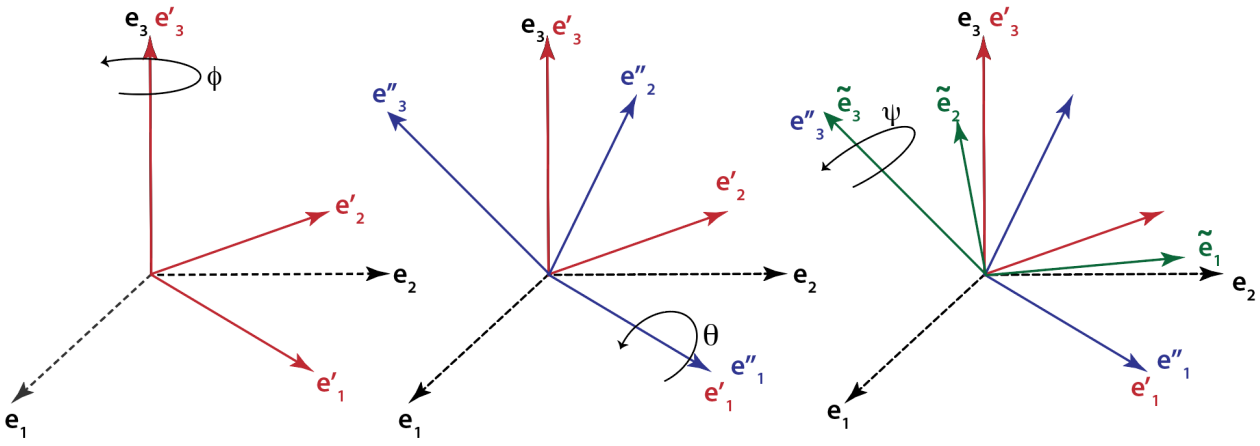


Figure 5.1. Specifying a body's orientation using Euler Angles

Thus, the rotation $(x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is represented by the following proper, or-

thogonal matrix

$$\mathsf{R} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\phi\sin\theta & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{5.2.5}$$

Its inverse (which is just the transpose) is:

$$\mathsf{R}^{-1} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \cos\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix} \tag{5.2.6}$$

5.3 Infinitesimal rotations and $SO(3)$ generators

Suppose **g** is some vector attached to a rigid-body rotating about an axis with unit vector **n**. In an infinitesimal time the vector **g** will have rotated by some infinitesimal angle $d\theta$ to a new vector $\mathbf{g}'$ where

$$\mathbf{g}' = \mathbf{g} + d\boldsymbol{\theta}\times\mathbf{g} = \begin{pmatrix} 1 & -n_3 d\theta & n_2 d\theta \\ n_3 d\theta & 1 & -n_1 d\theta \\ -n_2 d\theta & n_1 d\theta & 1 \end{pmatrix}\mathbf{g},\ d\boldsymbol{\theta} = d\theta\mathbf{n} \tag{5.3.1}$$

Letting $d\boldsymbol{\theta}_1$ and $d\boldsymbol{\theta}_2$ be two successive rotations then to first order

$$\mathbf{g}'' = \mathbf{g}' + d\boldsymbol{\theta}_2\times\mathbf{g}' = \mathbf{g} + (d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2)\times\mathbf{g} + o(d\theta^2) \tag{5.3.2}$$

implying that rotations about the same axis commute. Consequently, letting $\mathbf{J} = (\mathsf{J}_1, \mathsf{J}_2, \mathsf{J}_3)$ be the generator of rotations then we see that

$$\mathsf{R}(d\boldsymbol{\theta}) = \mathbb{1} + d\theta\mathbf{n}\cdot\mathbf{J} \tag{5.3.3}$$

we see that orthogonality requires

$$\mathsf{R}^T(d\boldsymbol{\theta})\mathsf{R}(d\boldsymbol{\theta}) = (\mathbb{1} + d\theta\mathbf{n}\cdot\mathbf{J}^T)(\mathbb{1} + d\theta\mathbf{n}\cdot\mathbf{J}) \tag{5.3.4}$$

$$= \mathbb{1} + d\theta\mathbf{n}\cdot(\mathsf{J}^T + \mathsf{J}) = \mathbb{1} \implies \mathsf{J}_i^T = -\mathsf{J}_i,\ i = 1,2,3 \tag{5.3.5}$$

Thus the generator of rotations must be anti-symmetric, and its most general form must therefore be

$$\mathbf{n}\cdot\mathbf{J} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \tag{5.3.6}$$

Comparison with (5.3.1) shows that

$$\mathbf{n}\cdot\mathbf{J} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \tag{5.3.7}$$

and since n_1, n_2, n_3 are independent we can define the following generators of rotations about the x_1, x_2, x_3 axes respectively

$$\mathsf{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathsf{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.3.8}$$

One can compute that these matrices satisfy the **SO(3) algebra**

$$[J_i, J_j] = \epsilon_{ijk} J_k \tag{5.3.9}$$

Note that we can build any rotation (not just infinitesimal) using the SO(3) generators. To see how, consider a rotation by an angle θ about the axis $\mathbf{n}$. Let us decompose this rotation into N infinitesimal rotations by an angle $d\theta = \frac{\theta}{N}$. Each of these infinitesimal rotations will take the form

$$\mathsf{R}(d\boldsymbol{\theta}) = \mathbb{1} + \frac{\theta}{N} \mathbf{n} \cdot \mathbf{J} \tag{5.3.10}$$

implying that

$$\mathsf{R}(\boldsymbol{\theta}) = \lim_{N \to \infty} \left(\mathbb{1} + \frac{\boldsymbol{\theta} \cdot \mathbf{J}}{N} \right)^N \tag{5.3.11}$$

Since successive, infinitesimal rotations commute, we can write this limit as an matrix exponential

$$\mathsf{R}(\boldsymbol{\theta}) = e^{\boldsymbol{\theta} \cdot \mathbf{J}} \tag{5.3.12}$$

5.4 Kinematics in rotating frames

Let $\mathbf{R}$ be the vector pointing to the rigid body's center of mass, $\mathbf{r}$ and $\mathbf{r}'$ denote the position of some P fixed in the rigid body relative to the lab frame S and the rigid body frame S' respectively. Now consider an infinitesimal segment of the rigid body's motion in which it rotates about an axis $\hat{\mathbf{n}}$ (through the CM) by an angle $d\theta$. This rotation produces a displacement $d\boldsymbol{\theta} \cdot \mathbf{r}$ of P in frame S'. The total displacement in S will thus be

$$d\mathbf{r} = d\mathbf{R} + d\boldsymbol{\theta} \times \mathbf{r}' \tag{5.4.1}$$

Dividing by dt we get a relation for the velocities of P in the two frames

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}' \tag{5.4.2}$$

where $\mathbf{V}$ is the CM velocity and $\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}$ is the angular velocity. This derivation worked so well because we implicitly set our axes' origin to lie on the center of mass about which P must rotate. Suppose we instead placed the origin of S' a distance $\mathbf{d}$ from the center of mass, forming a new reference frame S''. The position vector $\mathbf{r}''$ of P in the new S'' frame will be $\mathbf{r}'' = \mathbf{r}' + \mathbf{d}$. Thus

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{d} + \boldsymbol{\omega} \times \mathbf{r}'' \tag{5.4.3}$$

which suggests that we redefine the CM velocity in S'' to be

$$\mathbf{V}' = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{d} \tag{5.4.4}$$

while the angular velocity remains the same $\boldsymbol{\omega} = \boldsymbol{\omega}'$. This is a crucial point, it tells us that the angular velocity of a rigid body is independent of where we place our frame of reference (as long as the frame rotates with it).

Suppose we now wish to describe the position vector $\mathbf{r}$ of some point P fixed in space (not the rigid body). Then we have

$$\mathbf{r} = r_i \mathbf{e}_i = r'_i \mathbf{e}'_i \tag{5.4.5}$$

Taking the time-derivative we see that

$$\mathbf{r}_i = \dot{r}_i \mathbf{e}_i = \dot{r}'_i \mathbf{e}'_i + r_i \dot{\mathbf{e}}'_i \tag{5.4.6}$$

We now define $\boldsymbol{\omega}$, the angular velocity of the rotating axes $\mathbf{e}'_i$ with respect to $\mathbf{e}_i$ to satisfy

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}_i \tag{5.4.7}$$

If we take $r_i = R_{ji}(t) r'_j$ then

$$\frac{d\mathbf{e}'_i}{dt} = \frac{dR_{ij}}{dt}\mathbf{e}_j = \frac{dR_{ij}}{dt} R_{kj} \mathbf{e}'_k \equiv \omega_{ac} \mathbf{e}_c \tag{5.4.8}$$

where we have defined $\omega_{ik} = \dot{R}_{ij} R_{kj}$. Note that ω_{ik} defines an anti-symmetric 2-form ω since

$$\omega_{ki} = \dot{R}_{kj} R_{ki} = \frac{d}{dt}(R_{kj} R_{ki}) - \omega_{ik} = -\omega_{ik} \tag{5.4.9}$$

since $R_{kj}(t) R_{ki}(t) = \delta_{ji}$. Due to this anti-symmetry, the dual vector [1] $\boldsymbol{\omega}$ of ω_{ij} is given by

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\omega_{jk} \implies \boxed{\boldsymbol{\omega} = \frac{1}{2}\epsilon_{ijk}\omega_{jk}\mathbf{e}_i} \tag{5.4.10}$$

It follows that

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \tag{5.4.11}$$

as desired. Consequently

$$\dot{\mathbf{r}} = \dot{r}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{r} \tag{5.4.12}$$

We define the velocity of the point P as seen from S to be

$$\left.\frac{d\mathbf{r}}{dt}\right|_S = \dot{r}_i \mathbf{e}_i \tag{5.4.13}$$

and similarly the velocity of P as seen from S' to be

$$\left.\frac{d\mathbf{r}}{dt}\right|_{S'} = \dot{r}'_i \mathbf{e}'_i \tag{5.4.14}$$

[1] see differential geometry in Mathematical methods volume

Consequently we see that

$$\left.\frac{d\mathbf{r}}{dt}\right|_{S'} = \left.\frac{d\mathbf{r}}{dt}\right|_{S} + \boldsymbol{\omega} \times \mathbf{r} \tag{5.4.15}$$

As expected, the difference in the measured velocities is equal to the relative velocity of the two frames. Differentiating (5.4.15) we find that the accelerations in S and S' are related as

$$\ddot{\mathbf{r}} = \ddot{r}'_i \mathbf{e}'_i + 2\dot{r}'_i \dot{\mathbf{e}}'_i + r'_i \ddot{\mathbf{e}}'_i \tag{5.4.16}$$

$$= \ddot{r}'_i \mathbf{e}'_i + 2\dot{r}'_i(\boldsymbol{\omega} \times \mathbf{e}'_i) + r'_i \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{e}'_i \tag{5.4.17}$$

$$= \ddot{r}_i \mathbf{e}'_i + 2\boldsymbol{\omega} \times (\dot{r}_i \mathbf{e}'_i) + \dot{\boldsymbol{\omega}} \times (r_i \mathbf{e}'_i) + r_i \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{e}'_i) \tag{5.4.18}$$

and thus we have found a relation between the accelerations in S and S'

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_{S} = \left.\frac{d^2\mathbf{r}}{dt^2}\right|_{S'} + 2\boldsymbol{\omega} \times \left.\frac{d\mathbf{r}}{dt}\right|_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{5.4.19}$$

Using Newton's second law, if the force experienced by P in frame S is $\mathbf{F} = m\left.\frac{d^2\mathbf{r}}{dt^2}\right|_S$ then it follows that

$$\mathbf{F}' \equiv m\left.\frac{d^2\mathbf{r}}{dt^2}\right|_{S'} = \mathbf{F} - \underbrace{2m\boldsymbol{\omega} \times \left.\frac{d\mathbf{r}}{dt}\right|_{S'}}_{\text{Coriolis force}} - \underbrace{m\dot{\boldsymbol{\omega}} \times \mathbf{r}}_{\text{Euler force}} - \underbrace{m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{Centrifugal force}} \tag{5.4.20}$$

We see that in the rotating frame, particle S must experience three additional forces to explain its motion. These are not physical forces, but merely an artifact of the fact that the rotating frame is not an inertial frame of reference, and thus an additional set of **fictitious forces** is required to agree Newton's second law.

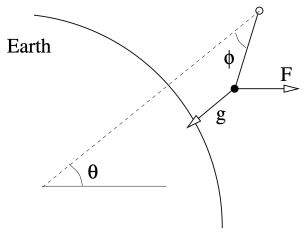
Centrifugal force

Not to be confused with the centripetal force which is very closely related to it, the centrifugal force is the fictitious force which replaces the centripetal force in a rotating frame.

To understand the necessity of this force, consider a person standing on a rotating carousel. From the lab frame's point of view, the forces acting on the person are the gravitational force pulling down, the normal force pushing up, and the friction force pulling in towards the carousel's center. The causes the person to move in a circular fashion with the carousel. From the person standing on the carousel's point of view, the non-fictitious forces acting on it are again the weight, normal force and friction, which alone would cause the person to drift towards the carousel's center. This description does not match what the observer sees, and indeed this is because there is an additional fictitious force, the centrifugal force which pushes outwards. Since from the person's point of view it is still relative to the carousel, the centrifugal force must be equal to the friction force (the centripetal force) in magnitude but opposite in direction:

$$\mathbf{F}_{\text{centrifugal}} = -\mathbf{F}_{\text{centripetal}} = m\omega^2 r \mathbf{e}_r \tag{5.4.21}$$

As a more interesting example, consider what happens when we suspend a pendulum on Earth at latitude $\pi - \theta$, as shown below There will be a gravitational force pointing radially



inwards as well as an oblique centrifugal force of magnitude

$$\mathbf{F} = -mR\omega^2\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{e}}_r) = -mR\omega^2\hat{\mathbf{z}} \times (\cos\theta\cos\varphi\hat{\mathbf{y}} - \cos\theta\sin\varphi\hat{\mathbf{x}}) \tag{5.4.22}$$

$$= mR\omega^2(\cos\theta\cos\varphi\hat{\mathbf{x}} + \cos\theta\sin\varphi\hat{\mathbf{y}}) \tag{5.4.23}$$

We can without loss of generality set our axes so that $\varphi = 0$ without loss of generality, so that **F** points in the x-direction, and the y-direction goes into the page. Then

$$\mathbf{F} = mR\omega^2\cos\theta(\cos\theta\hat{\mathbf{e}}_r - \sin\theta\hat{\mathbf{e}_\theta}) \tag{5.4.24}$$

Consequently Newton's second law implies that

$$m\omega^2 R\cos\theta\sin\theta = T\sin\phi \text{ and } m\omega^2 R\cos^2\theta = mg - T\cos\phi \tag{5.4.25}$$

The second equation implies that $T = \frac{mg}{\cos\phi} - m\omega^2 R\frac{\cos^2\theta}{\cos\phi}$ and thus

$$mg\tan\phi = m\omega^2 R(\cos^2\theta\tan\phi + \cos\theta\sin\theta) \implies \tan\phi(mg - m\omega^2 R\cos^2\theta) = m\omega^2 R\cos\theta\sin\theta \tag{5.4.26}$$

The angle ϕ the pendulum makes with the radial line is therefore

$$\tan\phi = \frac{\omega^2 R\cos\theta\sin\theta}{g - \omega^2 R\cos^2\theta} \tag{5.4.27}$$

Coriolis force

To understand the origin of this fictitious force, consider an observer standing on a carousel rotating with angular frequency ω. The observer starts to walk tangentially at constant radius R around the carousel with speed V. From the laboratory POV, the observer is moving at speed $v + \omega R$, so the friction force acting on their feet must be the centripetal contribution

$$\mathbf{F}_{\text{friction}} = -\frac{m(V + \omega R)^2}{R}\hat{\mathbf{e}}_r = -\left(\frac{mV^2}{R} + m\omega^2 R + 2m\omega V\right)\hat{\mathbf{e}}_r \tag{5.4.28}$$

Now consider the same situation from the observer's point of view. The friction force acting on them (which is the same in both frames) does not fully cancel out with the cen-

trifugal force $m\omega^2 R$ so that

$$\underbrace{\frac{mV^2}{R}}_{ma} \neq \underbrace{m\omega^2 R}_{F_{\text{centrifugal}}} - F_{\text{friction}} \tag{5.4.29}$$

We are therefore in need of an extra fictitious force $\mathbf{F}_{\text{coriolis}} = 2m\omega V\hat{\mathbf{e}}_r$ in order for Newton's equations to give the correct acceleration.

Euler force

To understand why this force is necessary, let's consider once more the rotating carousel. Assume the carousel has angular acceleration $\dot{\omega}$. Then in the lab frame, if the observer on the carousel is to remain fixed to it, there must be an additional friction force $F_{\text{friction}} = mR\dot{\omega}$. In the observer's frame this is not accounted for, just like in the centrifugal force's case.

5.5 Motion on Earth

There are several interesting consequences of the fact that we live on a rotating planet, and thus in a non-inertial frame of reference where fictitious forces play non-intuitive tricks.

Deflection of falling object

Consider an object

Cyclones

Foucault's pendulum

5.6 Lagrangian in rotating frames

5.7 Angular velocity using Euler-angles

We can write the angular velocity using the Euler angle parametrisation. Consider a rigid-body which in time dt moves from (ψ, θ, ϕ) to $(\psi + d\psi, \theta + d\theta, \phi + d\phi)$. By definition we must have that

$$\boldsymbol{\omega} = \dot{\psi}\mathbf{e}_3'' + \dot{\theta}\mathbf{e}_1' + \dot{\phi}\mathbf{e}_3 \tag{5.7.1}$$

where $\mathbf{e}_3, \mathbf{e}_1', \mathbf{e}_3''$ are shown in Figure 5.1. Using the rotation matrices, $\mathbf{e}_3$ is given in terms of $\mathbf{e}_i''$ by looking at the third column of R, and similarly $\mathbf{e}_1'$ by looking at the first column of $R(\psi)$ so that

$$\mathbf{e}_3 = \sin\theta\sin\psi\mathbf{e}_1'' + \cos\psi\sin\theta\mathbf{e}_2'' + \cos\theta\mathbf{e}_3'' \tag{5.7.2}$$

$$\mathbf{e}_1' = \cos\psi\mathbf{e}_1'' - \sin\psi\mathbf{e}_2'' \tag{5.7.3}$$

Similarly, in the lab frame we get

$$\mathbf{e}_3'' = \sin\theta\sin\phi\mathbf{e}_1 - \sin\phi\sin\theta\mathbf{e}_2 + \cos\theta\mathbf{e}_3 \tag{5.7.4}$$

$$\mathbf{e}_1' = \cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2 \tag{5.7.5}$$

Defining

$$\boldsymbol{\omega} = \omega_x \mathbf{e}_1 + \omega_y \mathbf{e}_2 + \omega_z \mathbf{e}_3 = \tilde{\omega}_x \tilde{\mathbf{e}}_1 + \tilde{\omega}_y \tilde{\mathbf{e}}_2 + \tilde{\omega}_z \tilde{\mathbf{e}}_3 \tag{5.7.6}$$

then we find that

$$\begin{cases} \tilde{\omega}_x = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \tilde{\omega}_y = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \tilde{\omega}_z = \dot{\psi} + \dot{\phi} \cos\theta \end{cases} \quad \text{and} \quad \begin{cases} \omega_x = \dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi \\ \omega_y = \dot{\theta} \sin\phi - \dot{\psi} \sin\theta \cos\phi \\ \omega_z = \dot{\phi} + \dot{\psi} \cos\theta \end{cases} \tag{5.7.7}$$

Rigid body dynamics

6

6.1 Rigid bodies

A rigid body is a body where the distance between its constituent particles does not vary. This body may be initially viewed as a discrete set of point-particles, and the passage to the continuous limit is achieved by replacing sums with integrals using a mass measure ρdV replacing the individual particle masses.

We will use two reference frames, an inertial lab frame fixed in space and a non-inertial frame fixed to the center of mass the of rigid body. Since the particles within the rigid body maintain the distance between them, a rigid body has six degrees of freedom, three that specify the position of the origin of the rotating frame relative to the lab, and three that specify the orientation of its axes relative to the lab. Indeed it is easy to convince yourself that given this data one could in principle draw a picture of the rigid body as seen from the lab frame (the Euler angles).

6.2 The Inertia tensor

In order to write the Lagrangian of a rigid body, and thus investigate its dynamics, we need to be able to write down its kinetic energy. Let the ith particle have mass $m^{(i)}$ and velocity $\mathbf{v}^{(i)}$ in the lab frame. From the previous chapter we know that

$$\mathbf{v}^{(i)} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}^{(i)} \tag{6.2.1}$$

where $\mathbf{V}$ is the CM velocity and $\mathbf{r}^{(i)}$ is the position of the ith particle in the rigid body frame. We then have that

$$T = \frac{1}{2}\sum_i m^{(i)}(\mathbf{v}^{(i)})^2 = \frac{1}{2}\sum_i m^{(i)}\mathbf{V}^2 + \sum_i \mathbf{V}\cdot(\boldsymbol{\omega}\times m^{(i)}\mathbf{r}^{(i)}) + \frac{1}{2}\sum_i m^{(i)}(\boldsymbol{\omega}\times\mathbf{r}^{(i)})^2 \tag{6.2.2}$$

For the middle term, we note that $\sum_i \mathbf{V}\cdot(\boldsymbol{\omega}\times m^{(i)}\mathbf{r}^{(i)}) = \mathbf{V}\cdot\left(\boldsymbol{\omega}\times\sum_i m^{(i)}\mathbf{r}^{(i)}\right) = 0$ since by definition the center of mass is set at the origin, so that $\mathbf{R} = \sum_i m^{(i)}\mathbf{r}^{(i)} = \mathbf{0}$. Consequently we are left with

$$T = \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\sum_i m^{(i)}\left[\omega^2(r^{(i)})^2 - (\boldsymbol{\omega}\cdot\mathbf{r}^{(i)})^2\right] \tag{6.2.3}$$

where $M = \sum_i m_i$ is the rigid-body's mass. The kinetic energy has been decomposed into a translational component due to the CM motion, and a rotational component. The latter can be written more succintly using index notation by noting that

$$m^{(i)}[\omega^2 (r^{(i)})^2 - (\boldsymbol{\omega} \cdot \mathbf{r}^{(i)})^2] = m^{(i)}[\omega_j \omega_j (r^{(i)})^2 - (\omega_j r_j^{(i)})(\omega_k r_k^{(i)})] \tag{6.2.4}$$

$$= \omega_j \omega_k m^{(i)}[\delta_{jk}(r^{(i)})^2 - r_j^{(i)} r_k^{(i)}] \tag{6.2.5}$$

Defining the **inertia tensor** I to be a rank-2 symmetric tensor with components

$$I_{jk} = \sum_i m^{(i)}[\delta_{jk}(r^{(i)})^2 - r_j^{(i)} r_k^{(i)}] \tag{6.2.6}$$

we then find that

$$T = \frac{1}{2} M \mathbf{V}^2 + \frac{1}{2} \boldsymbol{\omega}^T \overleftrightarrow{\mathsf{I}}\, \boldsymbol{\omega} \tag{6.2.7}$$

Passing to the continuous case is not difficult, one replaces the sum over particles with an integral

$$I_{jk} = \int \rho [\delta_{jk}(r^{(i)})^2 - r_j^{(i)} r_k^{(i)}] dV \tag{6.2.8}$$

As a matrix, $\overleftrightarrow{\mathsf{I}}$ can be written as

$$\overleftrightarrow{\mathsf{I}} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix} \tag{6.2.9}$$

Moreover, note that since $\overleftrightarrow{\mathsf{I}}$ is a second-rank *symmetric* tensor, it can be diagonalised by a special choice of axes, known as **principal axes of inertia**. These can be found by solving the eigenvector equation

$$\overleftrightarrow{\mathsf{I}}\, \boldsymbol{\eta}_j = I_j \boldsymbol{\eta}_j \tag{6.2.10}$$

Here the eigenvalues I_j are the principal moments of inertia and $\boldsymbol{\eta}_j$ are the vectors yielding the principal axes of inertia (in the basis in which $\overleftrightarrow{\mathsf{I}}$ was expressed originally). Letting $\mathsf{D} = (\boldsymbol{\eta}_1\ \boldsymbol{\eta}_2\ \boldsymbol{\eta}_3)$ then one finds that

$$\overleftrightarrow{\mathsf{I}}_{\text{principal}} = \mathsf{D}^T \overleftrightarrow{\mathsf{I}}\, \mathsf{D} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{6.2.11}$$

The calculation of the principal moment of inertia is greatly facilitated if the rigid-body is symmetric. Indeed, suppose the body has a plane of symmetry passing through the origin. For simplicity let this plane be the xy-plane. Then we have that $\rho(x,y,z) = \rho(x,y,-z)$ and thus by symmetry

$$I_{xz} = -\int \rho xz\, dxdydz = 0 \tag{6.2.12}$$

Similarly one finds that $I_{yz} = 0$ and so

$$\overleftrightarrow{\mathsf{I}} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \tag{6.2.13}$$

which clearly has an eigenvector

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies z\text{-axis is a principal axis} \tag{6.2.14}$$

So we see that the vector perpendicular to the plane of symmetry is a principal axis of inertia.

We can use this result to prove a more powerful theorem. Suppose a rigid body has a symmetry axis (it is a solid of revolution) through the origin, and consider two planes containing this axis. Then the vectors perpendicular to these planes, which can be chosen arbitrarily as long as they are mutually orthogonal, will also be principal moments of inertia. Setting z to lie on the symmetry axis, and x, y-axes to be in the plane normal to $\hat{\mathbf{z}}$. Moreover, by symmetry $\rho(x, y, z) = \rho(y, x, z)$ so we have that

$$I_{xx} = -\int \rho(y^2 + z^2)\, dxdydz = -\int \rho(x^2 + z^2)\, dx\, dy\, dz = I_{yy} \tag{6.2.15}$$

where we changed coordinates $x \mapsto y$. In the case of a planar lamina then

$$I_z = \int \rho(x^2 + y^2)\, dxdy = I_x + I_y \tag{6.2.16}$$

which is known as the **perpendicular axis theorem**. If the lamina further has rotational symmetry about the z-axis then letting $I_x = I_y = I$ we get that $I_z = 2I$. To summarise:

Symmetry and $\overleftrightarrow{\mathsf{I}}$: if a rigid body has a plane of symmetry, the vector perpendicular to this plane is a principal axes. Any two mutually orthogonal vectors in the symmetry plane are also principal axes with the same moments of inertia.

In our derivation of the inertia tensor, we assumed that the origin of S' was the center of mass. If instead we decided to place it a distance **d** away from the center of mass then by substituting $x'_i = x_i + d_i$ one easily finds that

$$I'_{jk} = I_{jk} + \mu(d^2\delta_{jk} - d_j d_k) \tag{6.2.17}$$

which is known as the **parallel axis theorem**.

From now on we will ignore translational degrees of freedom unless otherwise stated, and concentrate on the purely rotational motion.

The inertia tensor also allows us to write the angular momentum about the center of mass

as

$$\mathbf{L} = \sum_i m^i \mathbf{r}^{(i)} \times \mathbf{v}^{(i)} = \mathbf{V} \cdot \sum_i m^{(i)} \mathbf{r}^{(i)} + \sum_i m^{(i)} \mathbf{r}^{(i)} \times (\boldsymbol{\omega} \times \mathbf{r}^{(i)}) \tag{6.2.18}$$

$$= \sum_i m^{(i)} [(r^{(i)})^2 \boldsymbol{\omega} - (\mathbf{r}^{(i)} \cdot \boldsymbol{\omega}) \mathbf{r}^{(i)}] \tag{6.2.19}$$

$$= \sum_i m^{(i)} [\delta_{jk} (r^{(i)})^2 - r_j^{(i)} r_k^{(i)}] \omega_k \mathbf{e}_j \tag{6.2.20}$$

$$\mathbf{L} = \overleftrightarrow{\mathsf{I}}\, \boldsymbol{\omega} \tag{6.2.21}$$

which casts the rotational kinetic energy into

$$T = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} \tag{6.2.22}$$

If we adopt the principal axes of inertia where $\mathbf{L} = I_1 \omega_1 \mathbf{e}_1 + I_2 \omega_2 \mathbf{e}_2 + I_3 \omega_3 \mathbf{e}_3$ then the kinetic energy takes the particularly simple form

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 \tag{6.2.23}$$

Let's assume the rigid body has an azimuthal symmetry such that $I_1 = I_2 \neq I_3$. Using the Euler-angle formalism with the Euler axes $\{\tilde{\mathbf{e}}_i\}$ aligned with the principal axes, then one finds that

$$T = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 \tag{6.2.24}$$

6.3 Equations of motion for rigid bodies

Having discussed the energetics of rigid bodies we can now attempt to derive a set of equations of motion. Since a rigid body has at most 6 degrees of freedom we must seek for a set of six equations. Three of these can come from applying Newton's second law (conservation of linear momentum) and another three can come from applying conservation of angular momentum. The latter can be derived by differentiating $\mathbf{L}$:

$$\frac{d\mathbf{L}}{dt} = \sum_i m^{(i)} \frac{d\mathbf{r}^{(i)}}{dt} \times \mathbf{v}^{(i)} + \sum_i m^{(i)} \mathbf{r}^{(i)} \times \frac{d\mathbf{v}^{(i)}}{dt} \tag{6.3.1}$$

$$= \sum_i m^{(i)} \mathbf{v}^{(i)} \times \mathbf{v}^{(i)} + \sum_i \mathbf{r}^{(i)} \times \frac{d\mathbf{p}^{(i)}}{dt} \tag{6.3.2}$$

$$= \sum_i \mathbf{r}^{(i)} \times \mathbf{f}_i \tag{6.3.3}$$

where $\mathbf{f}_i$ is the net force acting on the ith particle. Note that if no external forces act on the rigid body then the torque due to internal forces will vanish by symmetry. Hence when computing $\sum_i \mathbf{r}^{(i)} \times \mathbf{f}_i$ we can neglect internal forces. Letting the **torque** about the center of mass be

$$\boldsymbol{\tau} = \sum_i \mathbf{r}^{(i)} \times \mathbf{f}^{(i)} \tag{6.3.4}$$

then we obtain

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \tag{6.3.5}$$

This law, which looks eerily similar to Newton's second law, expresses the conservation of angular momentum. Using this law and the inertia tensor, we can find three equations for the rotational degrees of freedom. For simplicity we align our axes with the principal axes of inertia $\{\mathbf{e}_i\}$. This means that the frame we are working in is non-inertial, and the time derivative of $\mathbf{L} = L_i\mathbf{e}_i$ will have two contributions since both L_i and $\mathbf{e}_i$ evolve through time. We find

$$\frac{d\mathbf{L}}{dt} = \frac{dL_j}{dt}\mathbf{e}_j + L_j\frac{d\mathbf{e}_j}{dt} = \frac{dL_j}{dt}\mathbf{e}_j + L_j(\boldsymbol{\omega}\times\mathbf{e}_j) \tag{6.3.6}$$

implying that

$$I_{jk}\frac{d\omega_k}{dt} + \epsilon_{jmn}\omega_m I_{nl}\omega_l = \tau_j \tag{6.3.7}$$

Here we used the fact that the principal moments of inertia are time-independent [1]. Now we use the fact that the inertia tensor is diagonal in our frame so $I_{nl} = I_n\delta_{nl}$ (no sum over n) so that $L_1 = I_1\omega_1, L_2 = I_2\omega_2, L_3\omega_3$. We find

$$I_j\frac{d\omega_j}{dt} + \epsilon_{jmn}\omega_m I_n\omega_n = \tau_j \tag{6.3.8}$$

or in component form

$$\begin{cases} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = \tau_1 \\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) = \tau_2 \\ I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = \tau_3 \end{cases} \tag{6.3.9}$$

These are known as **Euler's equations**.

We could have derived the law of conservation of angular momentum using the Lagrangian method too. Letting

$$L = \frac{1}{2}\mathbf{L}\cdot\boldsymbol{\omega} - U \tag{6.3.10}$$

then

$$\frac{\partial L}{\partial\boldsymbol{\omega}} = \mathbf{L},\ \frac{\partial L}{\partial\boldsymbol{\theta}} = -\frac{\partial U}{\partial\boldsymbol{\theta}} \equiv \boldsymbol{\tau} \tag{6.3.11}$$

The Euler-Lagrange equations then yield the conservation of angular momentum. To understand why we can identify $-\frac{\partial U}{\partial\theta}$ with the torque, note that the the potential energy change due to a system of particles each moving a distance $\delta\mathbf{r}^i$ under the influence of a net force $\mathbf{f}^{(i)}$ is

$$\delta U = \sum_i \mathbf{f}^{(i)}\cdot\delta\mathbf{r}^{(i)} = \sum_i \mathbf{f}^{(i)}\cdot(\boldsymbol{\theta}\times\mathbf{r}^{(i)}) = \delta\boldsymbol{\theta}\cdot\sum_i \mathbf{r}^{(i)}\times\mathbf{f}^{(i)} \implies \frac{\partial U}{\partial\boldsymbol{\theta}} = \boldsymbol{\tau} \tag{6.3.12}$$

[1] this is true for any frame rotating with the body, not just the principal axes

6.4 Rolling coin

We begin by discussing the motion of a coin of mass m and radius R on a plane inclined at an angle α to the horizontal. Throughout its motion the coin can be assumed to remain upright (perpendicular to the table). By symmetry, we see that the coin has a principal axis perpendicular to its face with $I_3 = \frac{mR^2}{2}$. By symmetry we can choose the other two principal axes to lie in the coin's plane, with $I_1 = I_2 = \frac{mR^2}{4}$ by the perpendicular axis theorem.

Letting x, y lie in the plane of the incline, the Lagrangian takes the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{mR^2}{8}(\omega_1^2 + \omega_2^2 + 2\omega_3^2) + mgy\sin\alpha \tag{6.4.1}$$

From the diagram we have that $\omega_3 = \dot{\phi}$ and $\omega_1^2 + \omega_2^2 = \dot{\theta}^2$ so

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{mR^2}{8}\dot{\theta}^2 + \frac{mR^2}{4}\dot{\phi}^2 - mgy\sin\alpha \tag{6.4.2}$$

We must also take into account the no-slip constraints

$$\dot{x} - R\dot{\phi}\sin\theta = 0, \text{ and } \dot{y} - R\dot{\phi}\cos\theta = 0 \tag{6.4.3}$$

which can be done using the method of Lagrange multipliers. We find

$$a_{1x} = 1, a_{1\theta} = 0, a_{1\phi} = -R\cos\theta, \ a_{2y} = 1, a_{2\theta} = 0, a_{2\phi} = -R\sin\theta \tag{6.4.4}$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{mR^2}{8}\dot{\theta}^2+\frac{mR^2}{4}\dot{\phi}^2+mgy\sin\alpha+\lambda_1(x-R\dot{\phi}\sin\theta)+\lambda_2(y-R\dot{\phi}\cos\theta) \tag{6.4.5}$$

The Euler-Lagrange equations read

$$\begin{cases} m\ddot{x} = \lambda_1, \ m\ddot{y} = \lambda_2 + mg\sin\alpha \\ \frac{mR^2}{4}\ddot{\theta} = 0 \\ \frac{mR^2}{2}\ddot{\phi} = -R(\lambda_1\sin\theta + \lambda_2\sin\theta) \end{cases} \tag{6.4.6}$$

From the third equation we quickly find that $\theta(t) = \theta_0 + \Omega t$. Also, combining the constraint equations with the first two equations of motion we get that

$$m\ddot{x} - mR(\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta) = \lambda_1 - mR(\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta) = 0 \tag{6.4.7}$$

$$m\ddot{y} - mR(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) = \lambda_2 + mg\sin\alpha - mR(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) = 0 \tag{6.4.8}$$

so that

$$\lambda_1 = mR(\ddot{\phi}\sin\theta + \dot{\phi}\Omega\cos\theta) \tag{6.4.9}$$

$$\lambda_2 = mR(\ddot{\phi}\cos\theta - \dot{\phi}\Omega\cos\theta) - mg\sin\alpha \tag{6.4.10}$$

We substitute these values of the Lagrange multipliers into (6.4.6) and find

$$\ddot{\phi} = -2\left(\ddot{\phi} - \frac{g}{R}\sin\alpha\cos\theta\right) \implies \ddot{\phi} = \frac{2g\sin\alpha}{3R}\cos(\theta_0 + \Omega t) \tag{6.4.11}$$

which can be integrated to yield

$$\phi = \phi_0 + \omega t - \frac{2g\sin\alpha}{3\Omega^2 R}\cos(\theta_0 + \Omega t) \tag{6.4.12}$$

Similarly, for x and y one finds that

$$\dot{x} = R\left(\omega + \frac{2g\sin\alpha}{3\Omega R}\cos\theta\right)\sin\theta \tag{6.4.13}$$

$$\dot{y} = R\left(\omega + \frac{2g\sin\alpha}{3\Omega R}\cos\theta\right)\cos\theta \tag{6.4.14}$$

$$\tag{6.4.15}$$

which can be integrated to yield

$$x = x_0 + \frac{g\sin\alpha}{3\Omega}t - \left[\frac{\omega R}{\Omega} + \frac{g\sin\alpha}{3\Omega^2}\sin(\theta_0 + \Omega t)\right]\cos(\theta_0 + \Omega t) \tag{6.4.16}$$

$$y = y_0 + \left[\frac{\omega R}{\Omega} + \frac{g\sin\alpha}{3\Omega^2}\sin(\theta_0 + \Omega t)\right]\sin(\theta_0 + \Omega t) \tag{6.4.17}$$

$$\tag{6.4.18}$$

6.5 Free spinning tops

Spinning disc

We consider a flat disc of mass m and radius R thrown almost horizontally. By symmetry, one principal axis will be normal to the frisbee with $I_3 = \frac{mR^2}{2}$ while the other two principal axes can be taken to lie inside the plane of the frisbee, with $I_1 = I_2 = I = \frac{I_3}{2}$. The Euler equations of motion are

$$I\dot{\omega}_1 + \omega_2\omega_3(I_3 - I) = 0 \tag{6.5.1}$$

$$I\dot{\omega}_2 + \omega_1\omega_3(I - I_3) = 0 \tag{6.5.2}$$

$$I_3\dot{\omega}_3 = 0 \tag{6.5.3}$$

From the last equation we immediately see that ω_3 is constant, so the "spin" of the frisbee will remain the same as expected from the conservation of angular momentum. We define

$$\Omega = \frac{I - I_3}{I}\omega_3 \tag{6.5.4}$$

so that the first two equations become

$$\begin{cases} \dot{\omega}_1 - \Omega\omega_2 = 0 \\ \dot{\omega}_2 + \Omega\omega_1 = 0 \end{cases} \implies \begin{cases} \ddot{\omega}_1 + \Omega^2\omega_1 = 0 \\ \ddot{\omega}_2 + \Omega^2\omega_2 = 0 \end{cases} \tag{6.5.5}$$

Therefore the angular velocity in the body frame is

$$\boldsymbol{\omega} = (\omega_0 \sin \Omega t, \omega_0 \cos \Omega t, \Omega) \tag{6.5.6}$$

To find the precession frequency we need to find what this vector looks like in the space frame. Comparing (6.5.6) with (5.7.7) we see that

$$\begin{cases} \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi = \omega_0 \sin \Omega t \\ \dot{\phi} \sin\theta \sin\psi - \dot{\theta} \sin\psi = \omega_0 \cos \Omega t \\ \dot{\psi} + \dot{\phi} \cos\theta = \omega_3 \end{cases} \tag{6.5.7}$$

To simplify matters we align the lab frame axes so that $\mathbf{L}$ lies along the z-axis. Then since the angle between $\omega_3\tilde{\mathbf{e}}_3$ and $\hat{\mathbf{z}}$, which is just θ, must be constant, we have $\dot{\theta} = 0$. Also one can see visually that $\dot{\psi} = \Omega$. Therefore squaring the first two equations and summing them we get

$$\dot{\phi} = \frac{\omega_0}{\sin\theta} = \frac{\omega_3 - \Omega}{\cos\theta} \tag{6.5.8}$$

and substituting $\Omega = \frac{I - I_3}{I}\omega_3$ we get

$$\dot{\phi} = \frac{I_3\omega_3}{I\cos\theta} \tag{6.5.9}$$

For a frisbee we have that $22I_3 = 2I$ and if $\theta \ll 1$ then the small-angle approximation implies that $\dot{\phi} \approx 2\omega_3$, so the wobble of the frisbee will have twice the frequency of its spin.

Part II

Fluid dynamics

Fluid properties

7

7.1 Simple fluids

As Batchelor puts it: "A" simple fluid is a material such that the relative positions of elements of the material change by an amount which is not small when suitable chosen forces, however small in magnitude, are applied to the material. That is, a **simple fluid** is a substance that flows (i.e. deforms continuously) under an applied force, irrespective of how small this force is. Such forces are known as shearing forces, they change the shape of the fluid without changing its volume. A fluid may resist this deformation, but this resistance cannot overcome the shearing force, so the "resisting force vanishes with the rate of deformation". Luckily, many fluids that we encounter daily such as water and air may be regarded as simple fluids quite accurately.

Simple fluids may also be understood on a microscopic scale. It is well known that two molecules in a substance interact following a van-der-Waals-like potential shown below:

Fluids and solid are known to have an average separation of order r_0, and thus strongly interact repulsively, while gas molecules have a separation of order $10r_0$, and thus very weakly interact attractively. So, while in statistical mechanics one can often model gases as perfectly non-interacting, in fluid mechanics we are not afforded the same privilege, interactions are key.

At this molecular scale the mass distribution is very uneven, with spikes centered near nuclei, and large regions of emptiness between molecules where the electronic wave-functions have decayed. Working with such non-uniform distributions is a hopeless task, so it is often helpful to regard the properties of a fluid as independent of the fact that the molecular distributions are quasi-discrete or continuous. This is known as the **continuum approximation**, and in the classical regime where low density quantum effects are unimportant it is very successful.

We are therefore allowed to consider a fluid as a mass distribution that is generally continuous in space and time.

7.2 Forces acting on fluids

We distinguish between two types of forces acting on matter. One type, known as **long-ranged**, slowly decrease as the separation of the interacting bodies is increased. As a result, long-ranged forces (or **volume forces**) are roughly constant over infinitesimal volume el-

ements $\delta^3\mathbf{r}$ and proportional to the size of this element dV. An example of a volume force is the gravitational force.

The second type, known as **short-ranged**, decrease rapidly as the separation of the interacting bodies is increased. They are only relevant when the separation is on the order of the molecular separation, and thus are appreciable when there is physical contact between the interacting elements. Therefore, short-ranged forces are exerted by a fluid element on another fluid element across their shared boundary surface. As a result, short-ranged forces (or **surface forces**) are roughly constant over an infinitesimal plane surface element $d^2\mathbf{r}$ [1] of the shared boundary, and proportional to its area dA. An example of a surface force is surface tension.

The total volume force acting on a fluid's volume element dV centered at $\mathbf{r}$ with density ρ at time t is:

$$d\mathbf{F}_{vol} = \mathbf{F}(\mathbf{r}, t)\rho dV \tag{7.2.1}$$

while the total surface force acting on a fluid's surface element dA centered at $\mathbf{r}$ with normal $\mathbf{n}$ at time t is:

$$d\mathbf{F}_{surf} = \boldsymbol{\Sigma}(\mathbf{r}, \mathbf{n}, t)dA \tag{7.2.2}$$

By convention, $\boldsymbol{\Sigma}$ is known as the **stress** exerted by the fluid lying on the surface element that $\mathbf{n}$ points to, on the fluid lying on the side that $\mathbf{n}$ points away from. By Newton's third law $\boldsymbol{\Sigma}$ is an odd function in $\mathbf{n}$, so $\boldsymbol{\Sigma}(-\mathbf{n}) = -\boldsymbol{\Sigma}(\mathbf{n})$.

Consider a tetrahedral volume element with orthogonal surfaces $\delta A_1, \delta A_2, \delta A_3$ oriented with unit normals $-\mathbf{x}, -\mathbf{y}, -\mathbf{z}$ respectively, and an inclined surface δA oriented with unit normal $\mathbf{n}$. Surface forces from the rest of the fluid will act on this tetrahedron across each of these surfaces:

$$\delta\mathbf{F}_{surf} = \boldsymbol{\Sigma}(\mathbf{n})\delta A + \boldsymbol{\Sigma}(-\mathbf{x})\delta A_1 + \boldsymbol{\Sigma}(-\mathbf{y})\delta A_2 + \boldsymbol{\Sigma}(-\mathbf{z})\delta A_3 \tag{7.2.3}$$

Using the relations $\delta A_1 = \mathbf{x}\cdot\mathbf{n}\delta A$,$\delta A_2 = \mathbf{y}\cdot\mathbf{n}\delta A$,$\delta A_3 = \mathbf{z}\cdot\mathbf{n}\delta A$ we find that:

$$(\delta\mathbf{F}_{surf})_i = [\boldsymbol{\Sigma}(\mathbf{n}) - (x_j\Sigma_i(\mathbf{x}) + y_j\Sigma_i(\mathbf{y}) + z_j\Sigma_i(\mathbf{z}))n_j]\delta A \tag{7.2.4}$$

Now the total forces acting on the tetrahedral volume element are a combination of body forces and surface forces:

$$(\delta m)\mathbf{a} = \delta\mathbf{F}_{vol} + \delta\mathbf{F}_{surf} \tag{7.2.5}$$

As we make $\delta V \to 0$, the LHS of (7.2.5) approaches zero, and so does the body force. Therefore we need $d\mathbf{F}_{surf} \to 0$ as $dV \to 0$:

$$\boldsymbol{\Sigma}_i(\mathbf{n}) = (x_j\Sigma_i(\mathbf{x}) + y_j\Sigma_i(\mathbf{y}) + z_j\Sigma_i(\mathbf{z}))n_j \tag{7.2.6}$$

This result tells us that the stress along a given normal can be decomposed into the stresses along three orthogonal directions. We define the **stress tensor**:

$$\Sigma_{ij} = x_j\Sigma_i(\mathbf{x}) + y_j\Sigma_i(\mathbf{y}) + z_j\Sigma_i(\mathbf{z}) \tag{7.2.7}$$

[1] technically this should be a volume element but the penetration depth of most surface forces is small

yielding:

$$\mathbf{\Sigma}_i = \sigma_{ij} n_j \tag{7.2.8}$$

The stress tensor σ_{ij} gives the ith component of the stress exerted across a surface element normal to the direction j. The diagonal elements σ_{ii} give the diagonal stresses (compression or expansion), while the off-diagonal elements σ_{ij} give the shearing stresses (deformation while maintaining volume).

We now take the stress torque on a volume element's boundary surface about a point $\mathcal{O}$ within this element:

$$\int (\mathbf{r} \times \boldsymbol{\sigma})_i \cdot d\mathbf{A} = \int \epsilon_{ijk} r_j \sigma_{kl} n_l dA \tag{7.2.9}$$

where $\mathbf{r}$ gives the displacement of the surface element $\mathbf{n}dA$ relative to our reference point $\mathcal{O}$. We can use the divergence theorem to simplify this integral:

$$\int \epsilon_{ijk} \frac{\partial r_j \sigma_{kl}}{\partial r_l} dV = \int \epsilon_{ijk} \left(\sigma_{kj} + r_j \frac{\partial \sigma_{kl}}{\partial r_l} \right) dV \tag{7.2.10}$$

We let the volume collapse onto $\mathcal{O}$ by keeping the boundary surface fixed, so that $V \to 0$. The total moment and the rate of change of angular momentum scale as $V^{4/3}$ while the first term scales as V. Therefore, we need this term to vanish:

$$\epsilon_{ijk} \sigma_{kj} = 0 \tag{7.2.11}$$

This relation establishes the symmetry of the stress tensor: $\sigma_{ij} = \sigma_{ji}$. It is well known that for any rank two symmetric cartesian tensor, three principal axes may be rotated so that this tensor becomes diagonal in this basis. In other words, we can always orient our axes so that the only stresses acting on the fluid are normal (such stress forces are known as **principal**), a fluid can always be regarded as in a state of tension/compression in three orthogonal directions.

Now consider a fluid of very small volume at rest. In the principal frame, the stress tensor is diagonal, and we write it in the form:

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \frac{1}{3}\sigma_{ii} \end{pmatrix} + \begin{pmatrix} \sigma_{11} - \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \sigma_{22} - \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \sigma_{33} - \frac{1}{3}\sigma_{ii} \end{pmatrix} \tag{7.2.12}$$

The first term is a uniform compressive (normal) stress. Thus it tends to change the volume of the fluid, and can be resisted. The second term however is a shearing stress, and thus cannot be resisted by a fluid since it maintains its volume, it only changes its shape. Consequently, for the fluid to be at rest we need the second term to vanish, making the principal stresses equal to each other:

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \frac{1}{3}\sigma_{ii} \end{pmatrix} \tag{7.2.13}$$

Note that rotating away from the principal axes will give the same matrix due to the

isotropicity of the principal stresses. As a result, , so only normal stresses are active! It is thus useful to define the **static fluid pressure** as the normal stress $p = -\frac{1}{3}\text{tr}\boldsymbol{\sigma} = -\frac{1}{3}\sigma_{ii}$ where the minus sign by convention means that positive pressure is compressive. Therefore

$$\sigma_{ij} = -p\delta_{ij} \tag{7.2.14}$$

The surface force across a plane element with normal $\mathbf{n}$ is given by $-p\mathbf{n}$.

7.3 Static equilibrium in fluids

The total force on a static fluid is a sum of the body and surface forces. Newton's first law then yields:

$$\int \rho\mathbf{F}dV - \int p\mathbf{n}dA = 0 \implies \int (\rho\mathbf{F} - \nabla p)dV = 0 \tag{7.3.1}$$

for any volume V contained within the fluid. This is only possible if the integrand vanishes inside the fluid, so the equilibrium condition in a fluid is:

$$\nabla p = \rho\mathbf{F} \tag{7.3.2}$$

For a fluid the shear stresses are not necessarily zero, so the equilibrium condition reads

$$\nabla p + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{r}} = 0 \tag{7.3.3}$$

Suppose that the body force per unit mass $\mathbf{F}$ is conservative, with a potential Φ so that $\mathbf{F} = -\nabla\Phi$. Then:

$$-\rho\nabla\Phi = \nabla p \implies (\nabla p) \times (\nabla\Phi) = 0 \tag{7.3.4}$$

so the unit normal vectors to the level surfaces of ρ and Γ and p must all be equal. Consider for example the case of a gravitational force where $\mathbf{F} = \mathbf{g} \implies \Phi = -\mathbf{g}\cdot\mathbf{r} = gz$. Then:

$$\nabla p = \rho\mathbf{g} \implies p_2 - p_1 = \rho\mathbf{g}\cdot(\mathbf{r}_2 - \mathbf{r}_1) \implies p_2 + \rho g z_2 = p_1 + \rho g z_1 \tag{7.3.5}$$

This is just an energy conservation law stating that a change in potential energy $\rho g z_2$ must come at the expense of a pressure difference (we can't have kinetic energies since we are at rest). This result also implies that there are no sudden jumps in pressure within a fluid or within the interface between two fluids. It follows that the surface of a fluid must be horizontal, since any variation in height would give a pressure difference.

Note that due to the arbitrariness of the reference point $\mathcal{O}$, we can in general write:

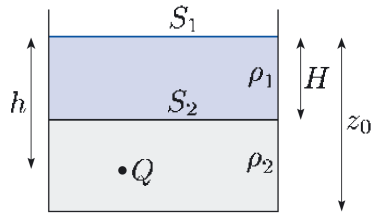
$$p(z) = c - \rho g z \tag{7.3.6}$$

where z is measured vertically upwards from the reference point, and $c = p(0)$ is a constant given by the pressure at $z = 0$. Suppose for example that the reference point is chosen to be on the interface between an incompressible fluid and the atmosphere. Then the pressure at a depth h in the fluid is given by:

$$p(h) = p_0 + \rho g h \tag{7.3.7}$$

where p_0 is the atmospheric pressure.

Example. An open vessel standing on a horizontal table contains a layer of an incompressible fluid of density ρ_1, floating on top of a layer of incompressible fluid of density ρ_2. Let S_1 be the surface separating fluid 1 and the atmosphere, and let S_2 separate fluid 2 from fluid 1. Suppose Q is a point at a depth h below the surface S_1. Let the reference point be



on S_2. Then for any h within fluid 2:

$$p_2 = p_1 + \rho_2 g(h - H) \tag{7.3.8}$$

but $p_1 = p_0 + \rho_1 g H$ so we have that:

$$p_2 = p_0 + \rho_1 g H + \rho_2 g(h - H),\ H \leq h \leq z_0 \tag{7.3.9}$$

We often refer to the surface between two fluids as a **free surface**.

Submerged solids

Suppose a flat plate with sides of length a, b is submerged in a fluid of density ρ so that its top edge is at a depth h_0. We let the top-left corner A of the plate be the origin of our coordinate frame. Consider an infinitesimal surface element $d\mathbf{A}$ with center $Q = (x, y)$. Since pressure acts uniformly on this static system, the force on this element will be $d\mathbf{f}(x, y) = -p(x, y)d\mathbf{A} = -(p_0 + \rho g(h_0 + x))d\mathbf{A}$. Integrating over the plate we have that:

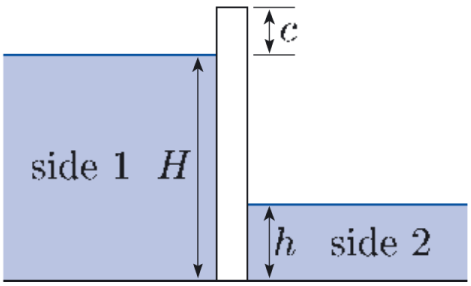
$$\mathbf{f} = -\mathbf{n}\int_0^b \int_0^a (p_0 + \rho g(h_0 + x))dxdy = -b(p_0 a + \frac{\rho g}{2}(2h_0 a + a^2)) \tag{7.3.10}$$

$$= -ab\left[p_0 + \rho g\left(h_0 + \frac{a^2}{2}\right)\right]\mathbf{n} \tag{7.3.11}$$

Note that this is just equal to the pressure p_M at the center of the plate times the area of the plate:

$$\mathbf{f} = -p_M A\mathbf{n} \tag{7.3.12}$$

An interesting application of this result are canal lock gates. One can use the difference in pressure between two bodies of water to block gates from opening in one direction. Suppose we have a fluid of total depth H on side 1 of a gate of height $H + c$, and another fluid of total depth h on side 2 of the gate. The gate has breadth b. Intuitively, we expect the gate to be locked in the direction side 2 $\rightarrow$ side 1. Indeed the total force on side 1 is



given by:

$$F_1 = \int_0^b \int_0^H [p_0 + \rho g(H - z)]dzdx + \int_0^b \int_H^{H+c} p_0 \tag{7.3.13}$$

$$= b(H + c)p_0 + \frac{1}{2}\rho g b H^2 \tag{7.3.14}$$

where we note that the force due to the fluid pressure is again equal to the bH times the pressure at the midpoint of the gate in contact with the fluid $\rho g \frac{H}{2}$. By the exact same calculations:

$$F_2 = \frac{1}{2}\rho g b h^2 + b p_0 (H + c) \tag{7.3.15}$$

giving a net force of:

$$F = \frac{1}{2}\rho g b (H^2 - h^2) \tag{7.3.16}$$

directed from side 1 to side 2. For $b = 3$m, $h = 2$m, $H =$5m, and if the fluid is water, then $F = 3 \times 10^5$N! If you ever find yourself in a sinking car, don't try to open the doors right away, wait for the water level to equalize and use the water pressure to your advantage.

Now suppose we fully submerge a cube of side length a so that its center of mass is at a depth h_0. The pressure force on the lateral sides of the cube cancel out. However, there will be a net force due to the height difference between the top and the base of the cube:

$$\mathbf{f} = \rho g a^2((h_0 + 2a) - h_0)\mathbf{z} = \rho g a^3 \mathbf{z} \tag{7.3.17}$$

where $\mathbf{z}$ points upwards.

This force is known as the buoyancy force, and is a general case of **Archimede's principle**: a body submerged in a liquid at rest will experience a resultant force directed upwards equal to the weight of the liquid it has displaced.

Indeed, suppose a body $\mathcal{V}$ with boundary surface $\mathcal{S}$ is submerged. Then:

$$\mathbf{f} = -\oint_{\mathcal{S}} p d\mathbf{S} \tag{7.3.18}$$

We also have that

$$M\mathbf{g} = \int_{\mathcal{V}} \rho \mathbf{g} dV = \int_{\mathcal{V}} \nabla p dV = \oint_{\mathcal{S}} p d\mathbf{S} = -\mathbf{f} \tag{7.3.19}$$

so we do indeed find that $\mathbf{f} = -M\mathbf{g}$.

Atmospheric pressure

Throughout our calculations we have dealt rather mysteriously with the atmospheric pressure p_0.

We cannot simply model the atmosphere as a fluid of constant density, since this would imply after a short calculation that the atmospheric pressure vanishes at a height of just 8.4km!

Let us model the atmosphere more realistically as an ideal gas with equation of state:

$$p = \rho RT \tag{7.3.20}$$

For an isothermal atmosphere, the temperature is constant implying $\frac{p}{\rho} = \frac{p_0}{\rho_0}$ where p_0 and ρ_0 are the atmospheric pressure and density at some reference point (e.g. sea level). Conesquently:

$$\frac{dp}{dz} = -\rho g = -\frac{p}{p_0}\rho_0 g \tag{7.3.21}$$

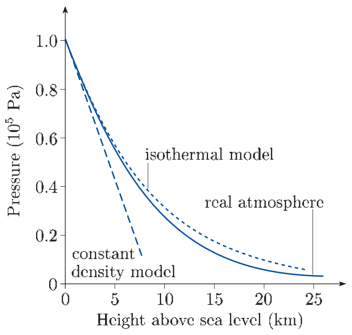
which can be integrated to give:

$$\ln p = -\frac{\rho g}{p_0} z + c \implies p(z) = Ae^{-\rho g z/p_0} \tag{7.3.22}$$

and using the condition $p(0) = p_0$ we find that:

$$p(z) = p_0 e^{-\rho g z/p_0} \tag{7.3.23}$$

The isothermal model is much more accurate than the constant density model, but still differs from experimental data at large heights. Suppose we now model the atmosphere



as a perfect gas satisfying the power law $p = k\rho^\gamma$ for some constants k, γ. Then:

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} \implies \rho = \rho_0 \left(\frac{p}{p_0}\right)^{1/\gamma} \tag{7.3.24}$$

Then:

$$\frac{dp}{dz} = k\gamma\rho^{\gamma-1}\frac{d\rho}{dz} = -\rho g \tag{7.3.25}$$

$$\implies \int \rho^{\gamma-2}d\rho = -\frac{gz}{k\gamma} \tag{7.3.26}$$

$$\implies \rho^{\gamma-1} = c - \frac{gz}{k\gamma}(\gamma-1) \tag{7.3.27}$$

Using the ideal gas law:

$$p = \rho RT \implies T(z) = \frac{k\rho^{\gamma-1}}{R} = A - \frac{(\gamma-1)g}{\gamma R}z \tag{7.3.28}$$

and using the condition $T(0) = T_0$ we find that:

$$T(z) = T_0 - \frac{(\gamma-1)g}{\gamma R}z \tag{7.3.29}$$

7.4 Liquid interfaces

We define an interface as the boundary between two media with distinct molecular structures. Consider an interface between a liquid and a gas. Recall from thermodynamics that a reversible isothermal process which increases the interface surface area by dA will require γdA work, where γ is known as the **surface tension**. The tensile force is then γdl along the perpendicular to $d\mathbf{l}$.

The tensile force acting on a surface $\mathcal{S}$ along a curve Γ is given by:

$$\mathbf{F}_\gamma = -\gamma \oint_\Gamma \mathbf{n} \times d\mathbf{l} \tag{7.4.1}$$

To see where this comes from, note that:

$$dW = d\mathbf{F}\cdot d\mathbf{x} = -\gamma(\mathbf{n}\times d\mathbf{l})\cdot d\mathbf{x} = -\gamma(d\mathbf{l}\times d\mathbf{x})\cdot\mathbf{n} = \gamma dA \tag{7.4.2}$$

as desired. Suppose the surface is defined by $\mathcal{S} : z - f(x,y) = 0$ giving a normal $\mathbf{n} = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$.

$$\mathbf{F}_\gamma = -\gamma \oint_\Gamma \Big(-\frac{\partial f}{\partial x}dy + \frac{\partial f}{\partial y}dx\Big) \approx \gamma\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big)dA \tag{7.4.3}$$

giving a pressure

$$p = \gamma\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial^2 y}\Big) = \gamma\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \tag{7.4.4}$$

Equilibrium is only achieved when this outwards pressure is countered by an equal inwards pressure, known as a Laplace pressure:

$$p = \gamma\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \tag{7.4.5}$$

We will discuss these effects due to surface tension more precisely in the next chapter.

Surface tension

8

8.1 Defining Surface Tension

It would be impossible for a person not to encounter the enthralling phenomenon that is *surface tension*. From taking a shower to washing hands with soap, and even the movement of foam when making tea, surface tension plays a central role, governing the dynamics and statics of interactions between fluids.

To understand how surface tension occurs, imagine looking at a water droplet under an immensely strong electron microscope, capable of clearly displaying individual water molecules. Firstly, consider the water molecules at the center of the droplet. These will feel cohesive forces from neighbouring water molecules. Hence, they will be "pulled" in all directions and will experience no net force. Let us now consider water molecules at the edge of the droplet, adjacent to the so-called "interface" surface (an imaginary surface delimiting two phases, such as water and air). Clearly, these molecules will experience both a cohesive force from neighbouring water molecules, but also adhesive forces from the nearby air molecules. Due to the imbalance between cohesive and adhesive forces, a net inward force will act on the outer layer of the droplet, giving it a spherical shape (see Figure 2.1)

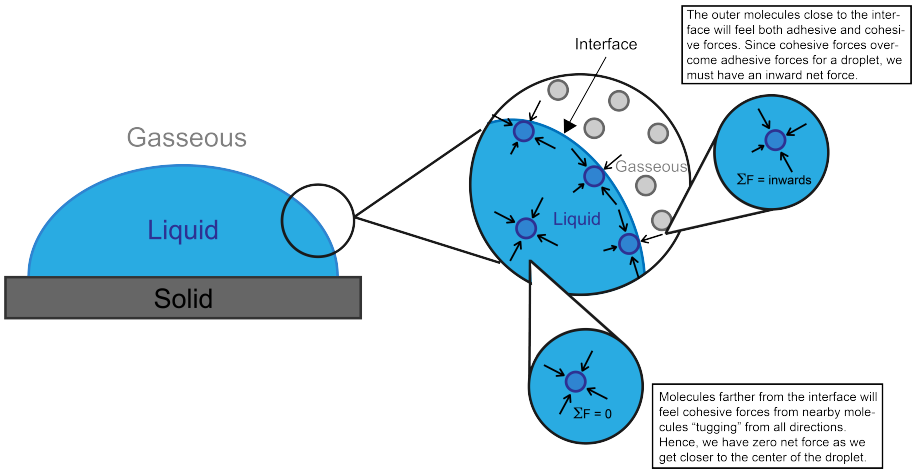


Figure 8.1. Diagram showing dynamics behind surface tension in a droplet

One might wonder why a droplet doesn't take a rectangular shape, or a pyramidal shape. The answer lies in surface optimization and the tendency of nature to minimize potential

energy. Indeed, surface tension γ has units $\frac{J}{m^2}$, energy per unit area. It is the energy needed to increase the droplet's surface area. If insufficient force acts on the droplet trying to increase its surface, then it will try to minimize it, "pushing" or "resisting" against any such force.

To better illustrate this idea, consider an arbitrary volume of water. We are asked to create a surface from that volume of water requiring the least effort. To do so, it is essential to note that the molecules forming this surface will oppose resistance to any increase in surface area due to cohesive forces, especially near the interface. In other words, the greater the surface area of this surface, the more work will be done to construct it. To use the least energy, the droplet will therefore have largest possible Volume-Surface ratio, trying to fit in the volume of water in the smallest possible surface area (the problem of defining a shape with the largest volume-surface ratio is known as the *Isoperimetric Inequality*). It can be proven that the sphere has the largest $V - S$ ratio. Any liquid will naturally rearrange itself into a sphere, as it requires the least energy (more formally, it has the least *surface energy*).

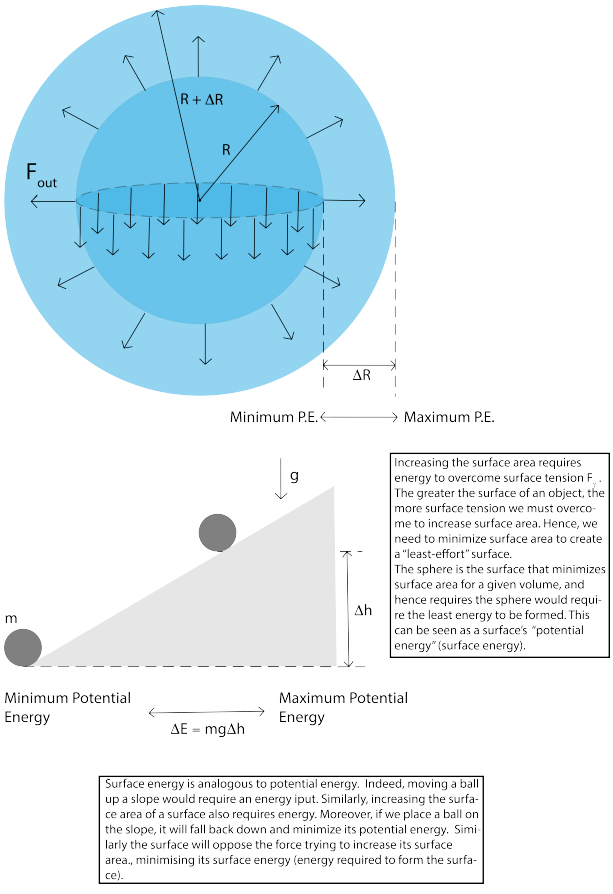


Figure 8.2. Analogy between surface energy and potential energy minimization.

By taking a spherical shape, the least work is required to increase surface area, the droplet essentially minimizes its surface energy (the energy needed to create the surface). This is

quite similar to a ball rolling up a hill. When we increase the surface area of a droplet, this requires energy input, just like trying to roll a ball up a slope. If we release the ball on this slope, it will try to minimize its potential energy, and fall back down. Analogously, the droplet will try minimizing its surface energy, resisting against any attempt at increasing surface area.

We can therefore define surface tension as the energy needed to increase the surface area of a liquid by unit area. It may also be considered as the force acting against deforming the surface per unit length. The higher the surface tension of a surface is, the less it is prone to increasing its surface area.

As hinted earlier, surface tension governs several phenomena we observe daily. As soon as we wake up, we usually go to the bathroom, and wash our face/hands. The reason we use soap is not only because of its antibacterial qualities, but also because it reduces surface tension of water. Thus, when coming into contact with our hands, water will be more easily deformed, entering into the crevices and wrinkles on our palms, and removing dirt. Another common way to observe surface tension is when making tea or coffee. Indeed, the reader has probably noticed the formation of foam and bubbles on the surface of the liquid. Usually, this foam collects either at the center of the surface, or at its circumference, for reasons we will allude to later on. The goal of this chapter will be to develop the physical laws describing surface tension, and related phenomena.

8.2 Deriving the Young-Laplace Equation

We shall now derive the Young-Laplace equation, without which most of the results in the rest of the chapter wouldn't be known. We will closely following the derivation given in *Siqvel and Skjæveland,* (2015). Surface tension can be defined as the force per unit length exerted on any fluid against increasing its surface area. Consider a curved surface, as shown below, representing the boundary between a liquid region and a gaseous region (known as the interface). Recall that when crossing this surface, there will be a change in pressure, called the Laplace Pressure (see Fig 1.3). As we can see, the internal pressure must counteract the outer atmospheric pressure, as well as the surface tension acting tangentially on its surface. This means that the pressure inside the droplet will obviously be larger than the outer pressure. Furthermore, this discontinuity in pressure when crossing the interface may also be seen as an explanation of why droplets try minimizing surface area. The internal pressure will keep increasing until it counteracts both the outward pressure and surface tension, which can be done by reducing the surface area of the droplet.

For an infinitesimal patch of this surface, the net force F_P caused by the pressure difference ΔP will be:

$$dF_P = \Delta P dS = \Delta P dL_1 dL_2 \tag{8.2.1}$$

We can then use the fact that $dL_1 = 2R_1 d\theta_1$ and $dL_2 = 2R_2 d\theta_2$:

$$dF_P = \Delta P dS = \Delta P (2R_1 d\theta_1)(2R_2 d\theta_2) \tag{8.2.2}$$

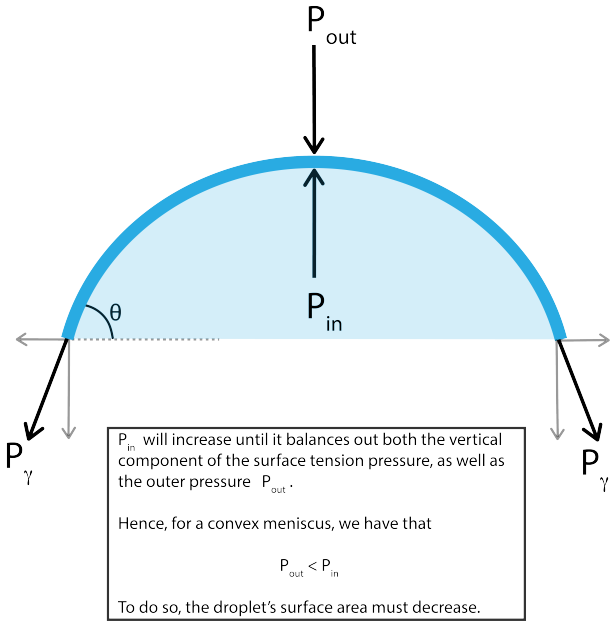


Figure 8.3. Pressure inside and outside of a spherical droplet or convex meniscus

Consider the surface tension force acting on the patch. Since we defined surface tension γ to be the force per unit length acting on the patch, then the total surface tensile force acting on it will be $\gamma \times$ infinitesimal length:

$$dF_\gamma^{(1)} = \gamma dL_2, \quad dF_\gamma^{(2)} = \gamma dL_1 \tag{8.2.3}$$

We can now take the total components acting vertically against F_P. Since there will be two forces of magnitude $dF_\gamma^{(1)}$ and two forces of magnitude $dF_\gamma^{(2)}$, we get that this component is:

$$2dF_\gamma^{(1)} \sin\theta_2 + 2dF_\gamma^{(2)} \sin\theta_1 \tag{8.2.4}$$

Recall that surface tension acts tangentially to the surface. Hence, $dF_\gamma^{(1)}$ acts on the principal line 2, and $dF_\gamma^{(2)}$ acts on the principal line 1.

We then use the small angle approximation $\sin x \approx x$ for simplification, and substituting our expressions for $dL_1, 2$:

$$dF_\gamma = \gamma(2(2R_1 d\theta_1)d\theta_2 + 2(2R_2 d\theta_2)d\theta_1). \tag{8.2.5}$$

For a static droplet, we must have that: $dF_P = dF_\lambda$. Hence:

$$\overbrace{\Delta P}^{Nm^{-2}} = \underbrace{\gamma}_{Fm^{-1}} \underbrace{\frac{R_1 + R_2}{R_1 R_2}}_{m^{-1}} \tag{8.2.6}$$

and finally we reach the Young-Laplace equation:

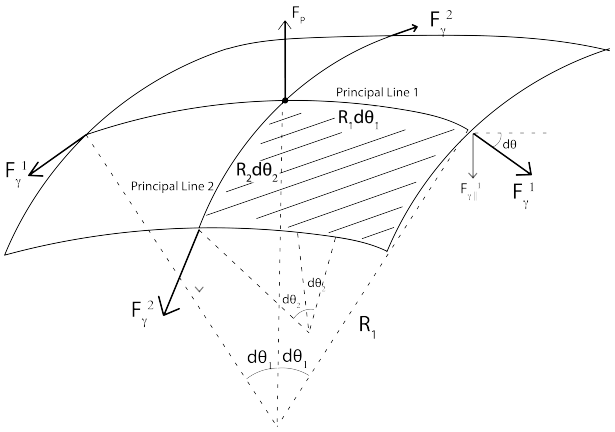


Figure 8.4. Infinitesimal patch over which surface tension acts along the edges

Young-Laplace Equation

$$\Delta P = \gamma\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \tag{8.2.7}$$

Let's stop a moment to analyze what this equation represents. The term on the right, ΔP, is the change in pressure when moving through the interface. On the right hand side, we have an expression with units $N \cdot m^{-2}$. This tells us that the more curved a surface is (the smaller $R_{1,2}$ are), the greater the Laplace pressure. This is expected, as we have a greater component due to surface tension against which internal pressure must act against. This widens the gap between pressure inside and outside.

Furthermore, as $R_{1,2} \longrightarrow \infty$ (we get a flat surface, a plane), we have that the Laplace pressure decreases very quickly $\Delta P \longrightarrow 0$. This makes sense, and agrees with the well known result that the pressure difference acting on a flat surface must be zero for equilibrium to be satisfied, a property that isn't necessarily true for curved surfaces as demonstrated.

We extend our formula further, using some notions of differential geometry. Note that the mean curvature of a surface at a point, H can be defined as the arithmetic mean of the minimum and maximum curvature (principal curvatures):

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{8.2.8}$$

where κ_1, κ_2, the principal curvatures of the surface. This then clearly yields for our patch:

$$H = \frac{1}{2}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \tag{8.2.9}$$

where, as discussed earlier, R_1 and R_2 are the radii of principal curvatures at point P.

$$\Delta P = \gamma(2H) \tag{8.2.10}$$

Using Frenet-Serret equations, we know that the mean curvature is:

$$H = -\frac{1}{2}\nabla \cdot \hat{\mathbf{n}} = -\frac{1}{2}|\nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right)| \tag{8.2.11}$$

which allows us to write more generally:

$$\Delta P = -\gamma(\nabla \cdot \hat{\mathbf{n}}) \tag{8.2.12}$$

This is a non linear partial differential equation, which relates the pressure difference through an interface and the shape of the interface surface. For positive curvature (e.g. concave meniscus), the Laplace pressure will be negative, whereas for negative curvature (e.g. convex meniscus), the Laplace pressure will be positive. Oddly, for a convex meniscus, the pressure outside is actually greater than the pressure just under the meniscus. As we will see later, the Young-Laplace equation is extremely powerful when solving problems for curved liquid surfaces. Three such problems are the floating body, the shape of the meniscus and the profile of a water droplet. These can all be solved by utilizing the Young-Laplace PDE.

Floating Bodies

We shall now adress the first problem introduced earlier, the statics of floating bodies, using the approach in *D. J. Vella*, (2007).

Consider an object (such as a metal pin, which is denser than water) of sufficiently small mass m "suspended" on a liquid. This object is not submerged, assuming the surface tension forces F_γ between the molecules of the liquid are strong enough not to let the interface surface rip. However, the object displaces water to its sides, and will therefore feel a buoyant force F_B.

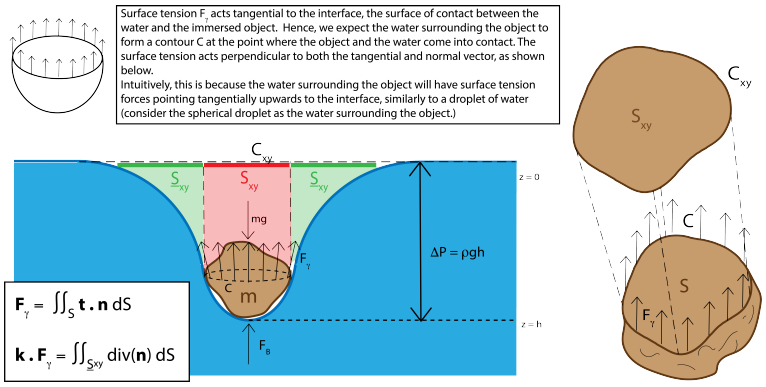


Figure 8.5. Body placed on a fluid depresses its surface, but doesn't necessarily sink due to surface tension and buoyancy.

Hence, for equilibrium we must have that $F_B + F_{\gamma\|} = mg$, where $F_{\gamma\|}$ is the component of F_γ acting vertically. Moreover, since pressure is defined as the force applied per unit

surface area, $F = -\iint_S PdA$. Using the Laplace-Young equation:

$$\mathbf{k} \cdot \mathbf{F}_\gamma = F_{\gamma\|} = -\gamma \int_{\bar{S_{xy}}} \nabla \cdot \hat{\mathbf{n}} \ dA \tag{8.2.13}$$

where S is the surface of contact between the floating body and the liquid, the interface surface. Furthermore, S_{xy} is the projection of S on the $x-y$ plane, and $\bar{S}_{xy}$ is $\mathbb{R}^2 \setminus S_{xy}$. We could also derive this result using the definition of surface tension as the force per length applied on the interface surface. Indeed, defining C as the contact line between the object and water expressed as an arc parametrized vector function $\mathbf{r} = \mathbf{r}(l)$, we get that:

$$\mathbf{F}_\gamma = \gamma \int_C \dot{\mathbf{r}} \times \hat{\mathbf{n}} \ dl \tag{8.2.14}$$

We took the cross product $\dot{\mathbf{r}} \times \hat{\mathbf{n}}$ because the surface tension force is orthogonal to both the vector tangent to **r** and the normal to the liquid surface **n**. Integrating over C then gives the length and direction over which surface tension acts. We can now evaluate the component of the tensile force acting vertically:

$$F_{\gamma\|} = \gamma \int_C \mathbf{k} \cdot (\dot{\mathbf{r}} \times \hat{\mathbf{n}}) \ dl \tag{8.2.15}$$

This is equivalent to projecting the contact line on the $x-y$ plane, forming C_{xy}, and then evaluating the component of the surface tension force acting vertically along $\hat{\mathbf{n}}$. This new path will have arc length l' and normal vector $\mathbf{n'}$ (this can be seen as transforming the integral using $l \longrightarrow l'$ and $\dot{\mathbf{r}} \times \mathbf{n} \longrightarrow \mathbf{n'}$) so that we get:

$$F_{\gamma\|} = \gamma \int_{C_{xy}} \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' \ dl' = \gamma \int_{C_{xy}} \hat{\mathbf{n}} \cdot d\mathbf{l}' \tag{8.2.16}$$

We can now use the two dimensional Divergence Theorem to simplify this integral.

> [2D Divergence Theorem] Let S be a region enclosed by a smooth curve ∂S, with normal vector $\hat{\mathbf{n}}$). Then, the following holds for any vector field **F** such that $\nabla \cdot \mathbf{F} \neq 0$:
>
> $$\iint_S \nabla \cdot \mathbf{F} \ dA = \int_{\partial S} \mathbf{F} \cdot d\hat{\mathbf{n}} \tag{8.2.17}$$

Since C_{xy} is simply the boundary of S_{xy}, we can define $\bar{S}_{xy}$ as $\mathbb{R}^2 \setminus S_{xy}$, with boundary $\bar{C}_{xy}$ We then have that:

$$\int_{C_{xy}} \hat{\mathbf{n}} \cdot d\mathbf{l}' = -\int_{C_{xy}} -\hat{\mathbf{n}} \cdot d\mathbf{l}' = -\int_{\bar{S}_{xy}} \nabla \cdot \hat{\mathbf{n}} \ dA \tag{8.2.18}$$

so that we finally reach:

$$F_{\gamma\|} = -\gamma \iint_{\bar{S}_{xy}} \nabla \cdot \hat{\mathbf{n}} \ dA \tag{8.2.19}$$

as required. By taking into account buoyant forces, we can then write by balancing forces:

$$-\iint_{\bar{S}_{xy}} \nabla \cdot \hat{\mathbf{n}} \; dA + \frac{F_B}{\gamma} = \frac{mg}{\gamma} \tag{8.2.20}$$

Consider now the two fluids separated by an interface. Denoting the liquid density ρ, the variation in vertical pressure between the two phases (liquid and gas) will be $\Delta P = \rho g h$, where h is the depression in the liquid. We can then write:

$$\iint_{\bar{S}_{xy}} \rho g h \; dA = mg - F_B \tag{8.2.21}$$

where we have used the Young-Laplace equation. Let us now evaluate F_B, the buoyant force acting on this object. We get using the Archimedean principle:

$$F_B = \rho g h S_{x,y} \tag{8.2.22}$$

Taking the ratio between the two yields:

$$\frac{F_B}{F_\gamma} = \frac{S_{xy}}{\bar{S}_{xy}} \tag{8.2.23}$$

This demonstrates that as the size of the object decreases, S_{xy} will also decrease, and hence this ratio will decrease. For smaller objects, such as needle pins, contrary to common belief, buoyancy is not the main reason they float, it is surface tension. Thus, we can conclude that objects with greater density than water can still float as long as they are small enough.

8.3 Capillary Action

Capillary action is another mechanism caused by surface tension, and leads to various interesting phenomena, such as the formation of menisci and the shape puddles and droplets.

When a liquid is placed in a narrow tube or cylinder of sufficiently small radius, this liquid may "rise" upwards. The adhesive forces overcome the cohesive forces, then the liquid molecules will be pulled by the walls of the container, rising.

Jurin's Law

Before we adress the meniscus problem introduced earlier, it is important to allude to the phenomenon of capillary action to truly understand the mechanism behind which menisci actually form, using two standard arguments to derive Jurin's Law.

Let us consider a cylinder filled with water, of radius R. Assuming that the meniscus has a spherical shape (we'll see later on how to more accurately define the profile of a meniscus) with contact angle θ with the cylinder's walls. It can be shown that the radius of curvature of the interface is $R \sec \theta$, so that the Laplace pressure is:

$$\Delta P = -\frac{2\gamma}{R \sec \theta} \tag{8.3.1}$$

Furthermore, the Laplace pressure can also be defined as:

$$\Delta P = P_{atm} - P_{men} = -\frac{2\gamma}{R\sec\theta} \tag{8.3.2}$$

Let us now consider two communicating vases as shown below.

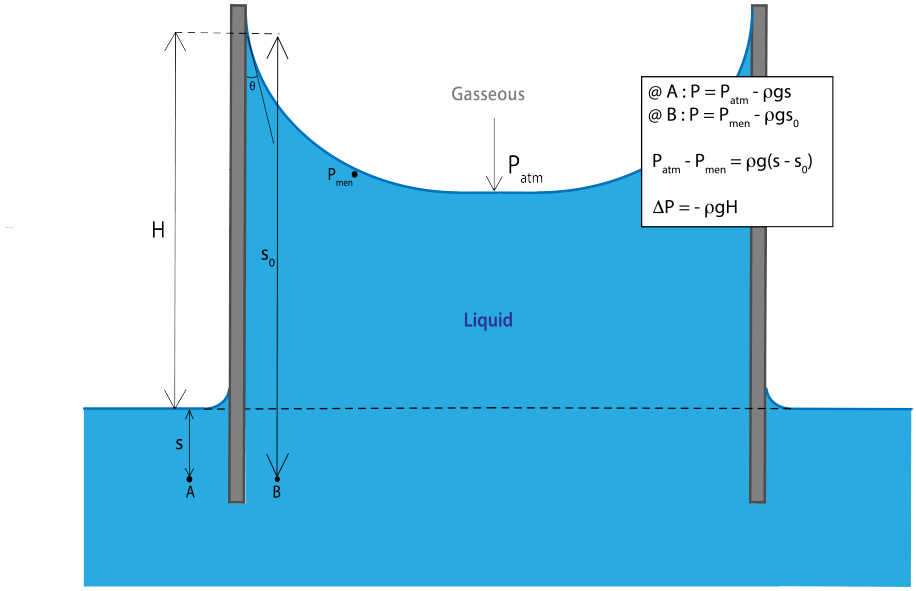


Figure 8.6. Capillary rise for a concave meniscus

Since pressure at equal heights in communicating vases must be equal, it follows that the pressure at an arbitrary height s under the water level is:

$$P_s = P_{atm} - \rho g s \tag{8.3.3}$$

for the "outer" vase. For the inner vase, the tube, we have that:

$$P_s = P_{men} - \rho g s_0 \tag{8.3.4}$$

where s_0 is defined as shown in Figure, P_{men} is the pressure at the meniscus. Equating these two expressions finally gives:

$$P_{atm} - P_{men} = \rho g(s - s_0) = -\rho g H \tag{8.3.5}$$

Using the Young-Laplace equation, we arrive at:

$$\rho_l g H = \frac{2\gamma}{R\sec\theta} \tag{8.3.6}$$

Rearranging we get Jurin's famous law for capillary rise

Jurin's Law

$$H = \frac{2\gamma \cos\theta}{R\rho g} \tag{8.3.7}$$

We could have also derived this result by equating forces as shown in *J. Pellicer et al.* (1995). The weight due to the column of water must be equal to the surface tension forces acting along the meniscus perimeter, causing the liquid to rise:

$$F_\gamma = 2\pi R\gamma \cos\theta = F_g = \pi\rho R^2 gH \tag{8.3.8}$$

Again, we should check boundary conditions to see if our answer makes physically sense. Jurin's law tells us that $H \propto \frac{1}{R\rho}$. The denser the liquid, the higher it will rise (there are some exceptions such as mercury). Moreover, the narrower the tube, the greater the rise. Both sound physically intuitive and are correct.

The Concave Meniscus ($\theta < \frac{\pi}{2}$)

Notice that the size of the meniscus plays a huge role in the derivation of Jurin's law. As seen earlier, the balance between cohesive and adhesive forces determines the shape of a meniscus (more specifically the contact angle formed with a wall, which defines the shape of a meniscus).

Consider once again a liquid placed in a tube. If liquid molecules are more attracted to the walls than to other liquid molecules (when adhesive forces overcome cohesive forces), one intuitively expects the meniscus to be concave. The molecules at the edges will be "dragged" upwards by adhesive forces, similar to a water column in a capillary tube would rise. Nearby molecules will move alongside as a result of cohesive forces. This may be easier to imagine if we interpret the cohesive forces as "chains": if the molecules at the edges move upwards, nearby molecules will move too. As we get farther from the tubes, these effects become more and more negligible, until they are null at the center of the container.

Instead, if the cohesive forces overcome the adhesive forces, then the meniscus will have a convex profile (similar to a droplet or puddle), since liquid molecules will try to "clump" all together, amassing near the center of the meniscus. Another major consequence is that we will have the "opposite" of capillary action, capillary fall. Instead of rising, the liquid molecules will try to "stick together", and actually fall (often observed in mercury). It follows that the equations governing puddles, droplets and menisci will be the same, as the conditions leading to their formation are identical.

Observe the right side of a meniscus in a cylinder containing a liquid columns. We will set $z = 0$ as the height at which the meniscus "converges" towards, and denote the contact angle with the wall of the cylinder as θ, as shown below.

Using (8.3.5), and following *Berg*, 2009, we can write that:

$$\gamma \nabla \cdot \hat{\mathbf{n}} = -\rho g z \tag{8.3.9}$$

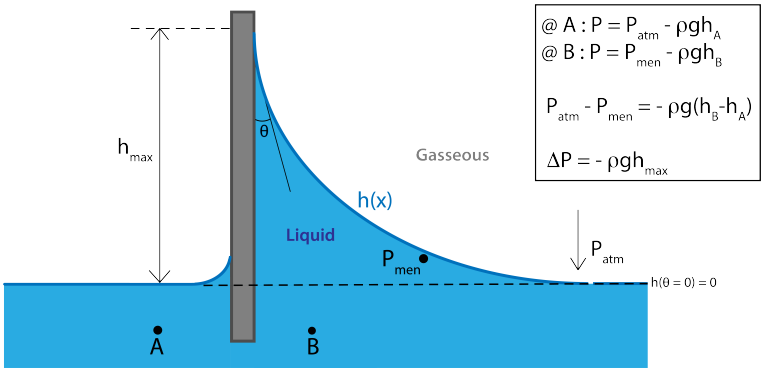


Figure 8.7. Profile of a concave meniscus

which can be used to define the shape of the meniscus. Indeed, if we define the surface of the meniscus as $f(x,z) = z - h(x)$, then:

$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\mathbf{z}} - h_x(x)\hat{\mathbf{x}}}{\sqrt{1 + h_x(x)^2}} \tag{8.3.10}$$

and we finally reach:

$$\nabla \cdot \hat{\mathbf{n}} = \frac{-h_{xx}(x)}{(1 + h_x(x)^2)^{\frac{3}{2}}} \approx -h_{xx}(x) \tag{8.3.11}$$

for $h_x(x) < 1$. Substituting into (8.3.9) we get the second order partial differential equation:

$$-\gamma \frac{\partial^2 h}{\partial x^2} = \rho g h, \quad h_x(0) = -\cot\theta, \tag{8.3.12}$$

whose solution is:

$$h(x) = \lambda \cot\theta e^{-\frac{x}{\lambda}} \tag{8.3.13}$$

where $\lambda = \sqrt{\frac{\gamma}{\rho g}}$ is the so called "Capillary length". This is another fundamental variable in the study of capillarity.[1] This expression gives the profile of a meniscus at a distance x from the wall of the container. As the capillary length increases, we expect that meniscus to be more curved (see next section), which agrees with our expression. A similar argument holds for the contact angle θ.

Some profiles for concave menisci at different contact angles are given.

Capillary Length λ and the shape of puddles

Consider a water droplet or puddle on a solid surface. We then know that the Laplace pressure at two points A, B inside this droplet, of radius of curvature R_A, R_B respectively

[1] We could have derived a similar result for a convex meniscus ($\theta > \frac{\pi}{2}$), by applying the initial condition $h_x(0) = \tan\theta$, giving as a solution:

$$h(x) = -\lambda \tan\theta e^{-\frac{x}{\lambda}} \tag{8.3.14}$$

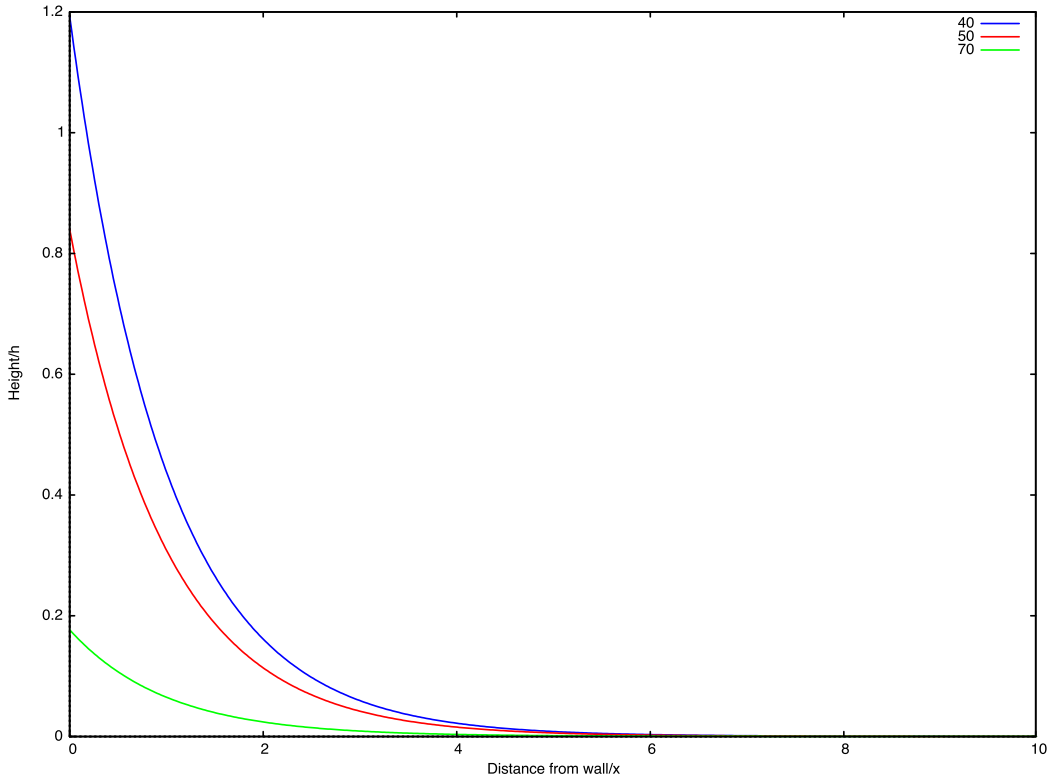


Figure 8.8. Menisci for contact angles $40°, 50°, 70°$, setting $\lambda = 1$

is then:

$$\Delta P_A = \frac{2\gamma}{R_A}, \quad \Delta P_B = \frac{2\gamma}{R_B} \tag{8.3.15}$$

We then have that the pressure difference between these two points is:

$$\Delta P_A - \Delta P_B = 2\gamma\Big(\frac{1}{R_A} - \frac{1}{R_B}\Big) \tag{8.3.16}$$

This is equal to the vertical hydrostatic pressure difference $\rho g h$, where h is the height difference between the A and B. Equating the two yields:

$$\Big(\frac{1}{R_A} - \frac{1}{R_B}\Big) = \frac{h}{2\frac{\gamma}{\rho g}} \tag{8.3.17}$$

Using dimensional analysis, we can conclude that the term $\frac{\gamma}{\rho g}$ must have units of $[\mathbf{L}]^2$, so that we may define the capillary length as:

$$\lambda = \sqrt{\frac{\gamma}{\rho g}} \tag{8.3.18}$$

The physical interpretation for this value is the distance over which a liquid-gas interface is curved. Thus, it follows that capillary length plays a vital role in determining the shape of a droplet or puddle. This is evident when analyzing cases where $h \gtrless \lambda$.

Case 1: $h > \lambda$

For droplets where $h > \lambda$, we then have using (8.3.17) that

$$\frac{1}{R_A} - \frac{1}{R_B} >> 0 \tag{8.3.19}$$

It follows that the radius of curvature at point B will be greater than at point A. This means that the curvature decreases as we move upwards, and hence we expect the top part of the droplet to be flat, and become more curved as we move downwards.

Case 2: $h < \lambda$

Using our physical intuition, for $h < \lambda$ we expect the droplet to have a spherical shape. Indeed, using the same procedure, we get that:

$$\frac{1}{R_A} - \frac{1}{R_B} \approx 0 \tag{8.3.20}$$

This means that the radius of curvature between any two arbitrary points A, B inside the droplet is the same. Hence, we must have a spherical droplet.

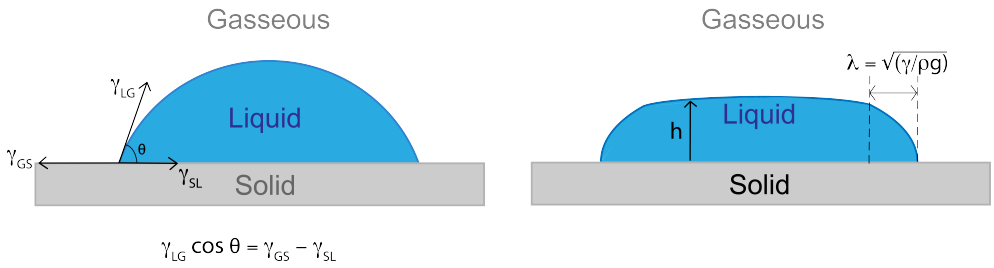


Figure 8.9. Comparison between puddles with $h > \lambda$ and $h < \lambda$.

One might now wonder how to calculate the maximum height of a droplet on an ideal smooth hydrophobic surface. We have that the net energy density or net force per unit length acting on the contour of the three phases(solid, liquid, gas), the interface, must be null. If we denote the surface tension of the solid-liquid, liquid-gas and gas-solid interface as $\gamma_{SL}\gamma_{LG}\gamma_{GS}$, and the contact angle as θ, then balancing force per unit length between these three phases:

$$\gamma_{SL} + \gamma_{LG}\cos\theta = \gamma_{GS} \tag{8.3.21}$$

which can be rearranged into the Young equation (not to be mistaken with the Young-Laplace equation):

Young Equation

$$\cos\theta = \frac{\gamma_{GS} - \gamma_{SL}}{\gamma_{LG}} \tag{8.3.22}$$

As the surface tension between the liquid and gas phases increases, the angle of contact must decrease. This agrees with Young's equation. Since surface tension is essentially how

much a surface pushes against increasing its surface area, if surface tension is greater, we expect its surface area to get smaller and smaller, causing a decrease in contact angle.

Going back to the derivation in subsection 2.3.2, we can write:

$$-\rho g x = \frac{\gamma g_{xx}(x)}{(1+g_x(x)^2)^{\frac{3}{2}}} \tag{8.3.23}$$

Substituting $q = g_x(x)$, we can solve this ODE:

$$-\frac{1}{2}\rho g h^2 = \frac{\gamma q}{\sqrt{1+q^2}} + C \tag{8.3.24}$$

$$= \gamma \cos\theta + C \tag{8.3.25}$$

We can now set initial conditions $h(\theta = 0) = 0$, so that $C = -\gamma$. Finally, we have the result:

$$h = \sqrt{\frac{2\gamma}{\rho g}(1-\cos\theta)} \tag{8.3.26}$$

which can be rewritten using Young's equation:

$$h = \sqrt{\frac{2}{\rho g}(\gamma - \gamma_{GS} + \gamma_{SL})} \tag{8.3.27}$$

The same result can be rewritten as

$$h = 2\lambda \sin\left(\frac{\theta}{2}\right) \tag{8.3.28}$$

Observing figure 2.9, notice that we can define a *spread factor* S as the difference between the surface energies trying to "spread" the droplet (pointing outwards), and the surface energies pointing inwards:

$$S = \gamma + \gamma_{SL} - \gamma_{GS} \tag{8.3.29}$$

so that we finally reach:

$$h = \sqrt{\frac{2S}{\rho g}} \tag{8.3.30}$$

This makes sense from a physical standpoint. Indeed, one would expect that for a puddle of droplet to have a greater maximum height, it would be "pushing inwards" more than it would be "pushing outwards". In other words, the greater the spreading parameter S is, the more spherical we'd expect the droplet to be. This agrees with (8.3.30). To conclude, we provide a table to summarize our results on menisci.

Concave Meniscus	Convex Meniscus
$h(x) \approx \lambda \cot\theta e^{-\frac{x}{\lambda}}$	$h(x) \approx -\lambda \tan\theta e^{-\frac{x}{\lambda}}$
$h_{max} = \sqrt{\frac{2\gamma}{\rho g}(1-\sin\theta)}$	$h_{max} = \sqrt{\frac{2\gamma}{\rho g}(1-\cos\theta)}$
$h_{max} \approx \lambda \cot\theta$	$h_{max} \approx \lambda \tan\theta$

Table 8.1. Table Summarizing Shape of Menisci

[2]

8.4 Minimal Surfaces

Consider a soap film produced when immersing a frame into a water-soap solution. Since we have no change in pressure when moving through the interface layer, it follows from Laplace's equation that the mean curvature of this soap film must be zero. Such types of surfaces that minimize surface area by having zero mean curvature at all points are called minimal surfaces. Indeed, *A. Presley* (2012) gives the definition of a minimal surface as:

A minimal surface is a surface whose mean curvature is zero everywhere.

They are given by the solutions to the *minimal surface equation*:

Minimal Surface Equation

$$\nabla \cdot \left(\frac{\nabla f}{(1+|\nabla f|^2)^{\frac{1}{2}}} \right) = 0 \tag{8.4.1}$$

Quite obviously, a simple plane would satisfy this equation. Another solution is the Helicoid, the second non-trivial solution to be discovered after the Catenoid (see Fig. 2.10).

Note that, except for the plane, all other solutions of the minimal surface equation will have non-zero curvature at some points. However, they average out at every point to be zero.

[2] We could have also derived the formula for a concave meniscus. Consider:

$$-\frac{1}{2}\rho g h^2 = \frac{\gamma q}{\sqrt{1+q^2}} + C = \gamma \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + C = \gamma \sin\theta$$

Using the initial condition that $h(q=\infty)=0$:

$$h = \sqrt{\frac{2\gamma}{\rho g}(1-\sin\theta)}$$

One could have also used trigonometric identities and have shown that:

$$h = \sqrt{\frac{2}{\rho g}(1-\cos\psi} = \sqrt{\frac{2}{\rho g}(1-\cos(90-\pi)} = \sqrt{\frac{2}{\rho g}(1-\sin\theta)}$$

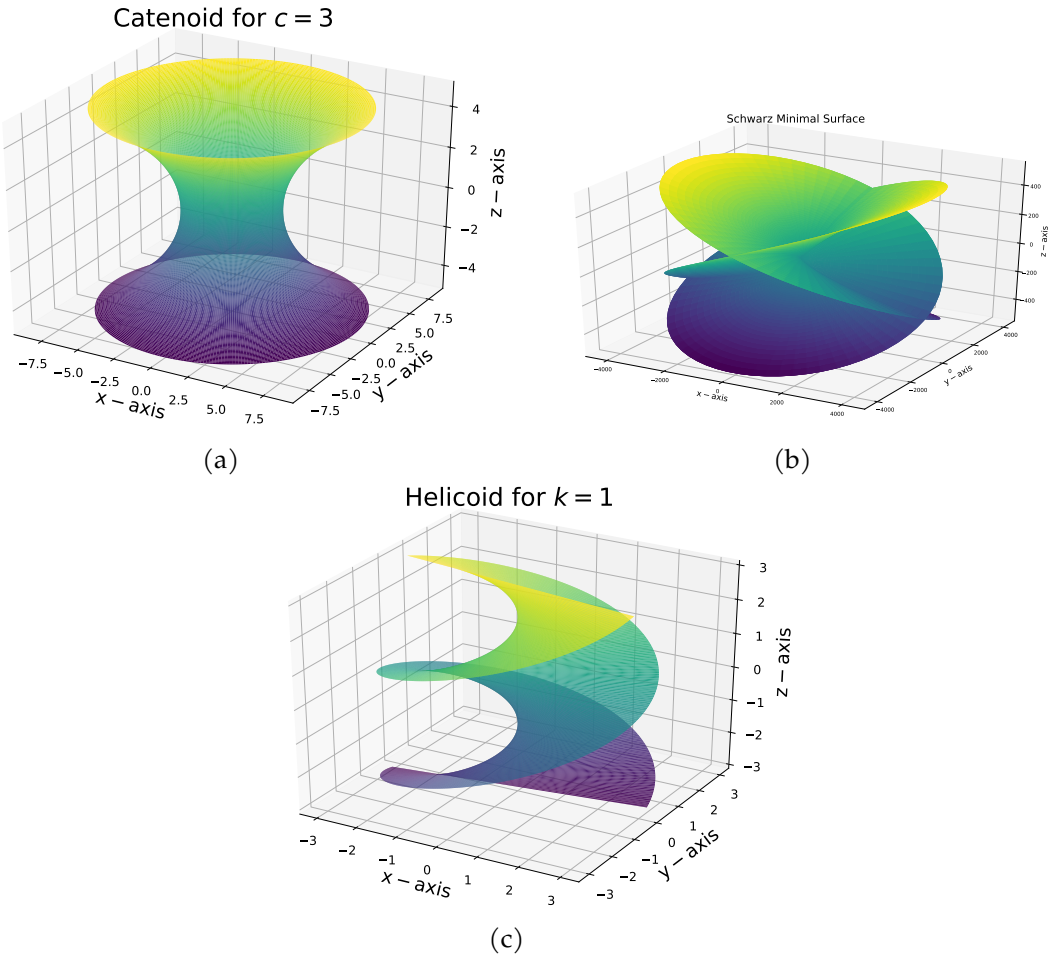


Figure 8.10. a) Plot of a Catenoid b) Plot of Henneberg Surface c) Plot of Helicoid

Moreover, all these surfaces have a "soap film frame", the frame that contains the set of all points the surface must contain, while still minimizing its surface area. For the Catenoid and the Helicoid, these frames are quite easy to imagine. The former is generated when immersing two elliptical rings parallel to each other in soap, whereas the latter is formed when using a helix.

Fluid kinematics

9

9.1 Describing moving fluids

We now turn our attention to the non-static case. We will deal with the distribution of the fluid velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$, which together with the fluid pressure $p(x, y, z, t)$ and density $\rho(x, y, z, t)$ fully determine all physical properties of the fluid. Indeed the fundamental equations of fluid dynamics which express conservation laws are expressed using these quantities.

9.2 Mass conservation

One important conservation law that applies to fluids is the local conservation of mass. Let us consider a region $\mathcal{V}$ of space, so that $\int_{\mathcal{V}} \rho \, d^3\mathbf{r}$ gives the total mass of the fluid contained within this region. We also have that $\rho\mathbf{v}$ represents the mass per unit area (cross-section) per unit time passing through this region, so that $\rho\mathbf{v} \cdot d\mathbf{A}$ gives the mass per unit time flowing through the surface element $d\mathbf{A}$. When integrated over $\partial\mathcal{V}$ this gives the total fluid mass flux through the region. For mass to be conserved in $\mathcal{V}$ we need the rate of change of the mass in $\mathcal{V}$ summed with the mass flux out of $\mathcal{V}$ to be equal to zero. Therefore

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \, d^3\mathbf{r} + \oint_{\partial\mathcal{V}} \rho\mathbf{v} \cdot d\mathbf{A} = 0 \tag{9.2.1}$$

We can use the Divergence theorem to write the surface integral as a volume integral

$$\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) \right) d^3\mathbf{r} = 0 \tag{9.2.2}$$

Since this must hold for any $\mathcal{V}$, the integrand has to vanish

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \tag{9.2.3}$$

This equation is known as the **continuity equation**, it states that any local change in density must be accounted for by a net divergence of mass flux. In other words fluid must be displaced out of a region for the mass within this region to change. Expanding the

divergence allows us to write the continuity equation as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0 \tag{9.2.4}$$

The quantity $\rho \mathbf{v}$ has played a fundamental role in this discussion and thus deserves its own name. It is known as the mass flux density $\mathbf{j}$, it is a vector field whose direction gives the direction of the fluid's motion, and whose magnitude yields the mass flux per unit time to a surface perpendicular to $\mathbf{v}$.

9.3 Velocity conservation

We again consider some arbitrary region $\mathcal{V}$ in the fluid. The total surface force acting on this volume is given by

$$\mathbf{F}_{\text{surf}} = -\oint_{\partial \mathcal{V}} p \, d\mathbf{A} = -\int_{\mathcal{V}} \nabla p \, d^3\mathbf{r} \tag{9.3.1}$$

so using Newton's second law one finds that

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = 0 \tag{9.3.2}$$

If we also had volume forces $\mathbf{f}_{\text{vol}}$ e.g. gravity then we would find

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{f}_{\text{vol}} \tag{9.3.3}$$

This is a conservation law! It says that any local change in the velocity of the fluid must be accounted for by a pressure gradient. Unlike previously however, we have a full derivative in time rather than a partial derivative. This poses some problems because in general $\mathbf{r}$ will be time dependent, and therefore $\mathbf{v} = \mathbf{v}(\mathbf{r}(t), t)$.

We reason as follows: what we are interested in is the change in velocity $d\mathbf{v}$ in an infinitesimal time interval dt. There will be two contributions, one due to the change $d\mathbf{r}$ in position during dt yielding a change in $\mathbf{v}$, and the other due to the change in time (at constant $\mathbf{r}$) directly/explicitly yielding a change in $\mathbf{v}$. The second contribution is easiest to work out, it is just $\frac{\partial \mathbf{v}}{\partial t}\big|_{\mathbf{r}} dt$. The first contribution, on the other hand, is given by

$$dx \frac{\partial \mathbf{v}}{\partial x} + dy \frac{\partial \mathbf{v}}{\partial y} + dz \frac{\partial \mathbf{v}}{\partial z} \tag{9.3.4}$$

which is just the inner product of the directional derivative and the displacement $d\mathbf{r}$: [1]

$$d\mathbf{r} \cdot \nabla \mathbf{v} \tag{9.3.5}$$

Therefore we find that

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} dt + (d\mathbf{r} \cdot \nabla)\mathbf{v} \tag{9.3.6}$$

and therefore

[1] recall that $\mathbf{n} \cdot \nabla f$ gives the change in f along $\mathbf{n}$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \tag{9.3.7}$$

We define $\frac{d}{dt}$ in this context to be the **convective derivative**, it yields the time derivative of anything as we travel along with the fluid. The convective derivative is extremely useful as any local conservation law can be written as $\frac{dA}{dt} = 0$.

Substituting (9.3.7) into (9.3.3) we then get Euler's equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = 0 \tag{9.3.8}$$

It is important to note that throughout this derivation we have not taken into account any sources for energy dissipation such as viscosity. Therefore Euler's equation only holds for ideal fluids, which are completely characterised by p and ρ. Moreover, since there is no heat exchange in the fluid the motion we have described is reversible and adiabatic, or in other words isentropic: $dS = 0$. Since the entropy is constant we can write

$$\frac{dS}{dt} = 0 \implies \frac{\partial S}{\partial t} + (\mathbf{v}\cdot\nabla)S = 0 \tag{9.3.9}$$

which, together with the continuity equation yields the **entropic equation of continuity**

$$\frac{d(\rho S)}{dt} = \frac{\partial(\rho S)}{\partial t} + \nabla\cdot(\rho S\mathbf{v}) = 0 \tag{9.3.10}$$

For isentropic processes the specific enthalpy h satisfies the thermodynamic relation $dh = Tds + \frac{dp}{\rho} = \frac{dp}{\rho}$. Therefore, if the flow is steady (so that we are at equilibrium), then

$$\nabla h = \frac{1}{\rho}\nabla p \tag{9.3.11}$$

This allows us to write Euler's equation as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla h = 0 \tag{9.3.12}$$

Yet another useful form of Euler's equation comes from applying the vector identity [2]

$$\frac{1}{2}\nabla v^2 = \mathbf{v}\times(\nabla\times\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v} \tag{9.3.14}$$

then Euler's equation takes the form (letting $\boldsymbol{\omega} = \nabla\times\mathbf{v}$)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(h + \frac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = 0 \tag{9.3.15}$$

[2] Here's the proof

$$[\mathbf{v}\times(\nabla\times\mathbf{v})]^i = \epsilon^{ijk}v_j\epsilon_{klm}\partial^l v^m = (\delta^i_l\delta^j_m - \delta^i_m\delta^j_l)v_j\partial^l v^m = \frac{1}{2}\partial^i(v_m v^m) - v_l\partial^l v^i \tag{9.3.13}$$

or, taking the curl of both sides

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \tag{9.3.16}$$

In the presence of an external body force **F** then Euler's equation takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(h + \frac{1}{2}v^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{F} \tag{9.3.17}$$

This equation holds for both compressible and incompressible fluids, but does not take into account viscous flow.

As with all PDEs, a particular solution can only be found if we are also given sufficient boundary and initial value conditions. For ideal fluids in contact with a solid surface with unit normal **n**, it is implicit that $\mathbf{n} \cdot \mathbf{v} = w$ where w is the speed of the surface. If instead we are dealing with two immiscible fluids forming a boundary, then we must require the pressure and $\mathbf{n} \cdot \mathbf{v}$ to be continuous across the boundary.

9.4 Energy conservation

The energy per unit volume of a fluid is given by

$$\frac{1}{2}\rho v^2 + \rho\epsilon \tag{9.4.1}$$

where ϵ is the specific internal energy. We are interested in the rate of change of this energy with respect to time

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) = \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial(\rho\epsilon)}{\partial t} \tag{9.4.2}$$

To compute the last partial derivative we use the First law of thermodynamics

$$d\epsilon = Tds - pdv = Tds + \frac{p}{\rho^2}d\rho \tag{9.4.3}$$

to yield

$$d(\rho\epsilon) = \rho T ds + \left(\epsilon + \frac{p}{\rho}\right)d\rho = \rho T ds + h d\rho \tag{9.4.4}$$

and thus

$$\frac{\partial(\rho\epsilon)}{\partial t} = h\frac{\partial \rho}{\partial t} + \rho T\frac{\partial s}{\partial t} \tag{9.4.5}$$

We can use the entropic continuity equation to write this as

$$\frac{\partial(\rho\epsilon)}{\partial t} = h\frac{\partial \rho}{\partial t} - \rho T(\mathbf{v} \cdot \nabla)s \tag{9.4.6}$$

Similarly, we can massage the rest of (9.4.2) by using the continuity equation and Euler's equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{1}{2}v^2\nabla\cdot(\rho\mathbf{v}) - \mathbf{v}\cdot\nabla p - \rho\mathbf{v}\cdot((\mathbf{v}\cdot\nabla)\mathbf{v}) \tag{9.4.7}$$

$$= -\frac{1}{2}v^2\nabla\cdot(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\nabla\left(\frac{1}{2}v^2 + h\right) + \rho T\mathbf{v}\cdot(\nabla s) \tag{9.4.8}$$

where we used $\nabla p = \rho\nabla h - \rho T\nabla s$. Finally, the condition for energy to be conserved becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) = -\left(\frac{1}{2}v^2 + h\right)\nabla\cdot(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\nabla\left(\frac{1}{2}v^2 + h\right) \tag{9.4.9}$$

or more simply

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) + \nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)\right] = 0} \tag{9.4.10}$$

This is yet another conservation law, it tells us that any change in the energy of the fluid contained within a region must be accounted for by an energy flux divergence.

9.5 Bernoulli's equation

In the case of steady flow (defined by a time-independent velocity field $\mathbf{v}(x, y, z)$), Euler's equations simplify a great deal into

$$\nabla\left(\frac{1}{2}v^2 + h\right) - \mathbf{v}\times\boldsymbol{\omega} = 0 \tag{9.5.1}$$

We dot to the left with $\mathbf{l}$, the unit vector tangent to the streamline at every point in the fluid, and using the fact that $\frac{\partial\mathbf{v}}{\partial t} = \frac{\partial h}{\partial t} = 0$ we then find that

$$\mathbf{l}\cdot\nabla\left(\frac{1}{2}v^2 + h\right) = 0 \implies \frac{d}{dl}\left(\frac{1}{2}v^2 + h\right) = 0 \tag{9.5.2}$$

or in other words

$$\frac{1}{2}v^2 + h = \text{cnst. along a streamline} \tag{9.5.3}$$

If we also add an external body force with potential Φ then we get

$$\boxed{\frac{1}{2}v^2 + h + \Phi = \text{cnst.}} \tag{9.5.4}$$

which is known as **Bernoulli's equation**, it applies along any given **streamline of a steady, inviscid fluid with a conservative body force acting on it**. If we further add the constraint of constant density then we end up with

$$\boxed{\frac{1}{2}v^2 + \frac{p}{\rho} + \Phi = \text{cnst.}} \tag{9.5.5}$$

9.6 Momentum conservation

We can try to combine the mass conservation and velocity conservation to obtain a momentum conservation law. In index notation we find that

$$\frac{\partial}{\partial t}(\rho v_i) = \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} \tag{9.6.1}$$

$$= -\rho v_k\frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - \frac{\partial(\rho v_k)}{\partial x_k}v_i \tag{9.6.2}$$

$$= -\frac{\partial}{\partial x_k}(\rho v_i v_k) - \delta_{ik}\frac{\partial p}{\partial x_k} = -\frac{\partial \sigma_{ik}}{\partial x_k} \tag{9.6.3}$$

where we used the stress tensor

$$\sigma_{ik} = \rho v_i v_k + \delta_{ik}p \iff \boldsymbol{\sigma} = \rho\mathbf{v}\otimes\mathbf{v} + p\mathbb{1} \tag{9.6.4}$$

The physical interpretation of this stress tensor is best understood by integrating (9.6.3):

$$\frac{\partial}{\partial t}\int \rho v_i d^3\mathbf{r} = -\oint \sigma_{ik} dA_k \tag{9.6.5}$$

so just like the static case the stress tensor component σ_{ik} gives the ith component of the momentum flux per unit time passing through an area perpendicular to the x_k-axis. This time however, the presence of dynamics added a velocity term $\rho\mathbf{v}\otimes\mathbf{v}$. We therefore see that

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot\boldsymbol{\sigma} = 0 \tag{9.6.6}$$

is a conservation law for momentum!

9.7 Circulation conservation and Vorticity

Let Γ be a closed contour within the fluid at some instant in time, and let

$$C = \oint_\Gamma \mathbf{v}\cdot d\mathbf{l} \tag{9.7.1}$$

be the velocity circulation around Γ. This curve will be composed of the fluid particles at some instant, so as the particles move the contour will also move with the fluid. Since we are interested in the time derivative of the circulation, it will be helpful to rewrite the line integral as

$$C = \lim_{N\to\infty}\sum \mathbf{v}\cdot\delta\mathbf{r}_k \tag{9.7.2}$$

It follows that

$$\frac{dC}{dt} = \lim_{N\to\infty} \sum_{k=0}^{N} \left[\frac{d\mathbf{v}_k}{dt} \cdot \delta\mathbf{r}_k + \mathbf{v}_k \cdot \frac{d(\delta\mathbf{r}_k)}{dt} \right] \tag{9.7.3}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} \left[\frac{d\mathbf{v}_k}{dt} \cdot \delta\mathbf{r}_k + \frac{1}{2}\delta(v_k^2) \right] \tag{9.7.4}$$

$$= \oint_\Gamma \frac{d\mathbf{v}}{dt} \cdot d\mathbf{l} \tag{9.7.5}$$

since the line integral of a full differential vanishes. Using $\frac{d\mathbf{v}}{dt} = -\nabla h$ we then get that

$$\frac{d}{dt}\left(\oint_\Gamma \mathbf{v} \cdot d\mathbf{l} \right) = -\oint_\Gamma (\nabla h) \cdot d\mathbf{l} \tag{9.7.6}$$

So we find that the velocity circulation is a conserved quantity

$$\frac{d}{dt}\left(\oint_\Gamma \mathbf{v} \cdot d\mathbf{l} \right) = \frac{d}{dt}\left(\int_\mathcal{S} \boldsymbol{\omega} \cdot d\mathbf{A} \right) = 0 \tag{9.7.7}$$

Note that we used Euler's equation to write $\frac{d\mathbf{v}}{dt}$ as a gradient, so this result again only holds for ideal fluids. The quantity $\boldsymbol{\omega}$ is known as **vorticity**.

The conservation of circulation also yields another important result. Consider pathline in a fluid on which the vorticity vanishes at some point. Suppose we draw an infinitesimal circular contour around this point, then by Kelvin's theorem as this contour moves along the pathline the vorticity will not change and be equal to zero. Importantly, in the case of steady flows where streamlines and pathlines coincide, we find that if the vorticity vanishes somewhere on a streamline then it is zero on the entire streamline.A flow where $\boldsymbol{\omega}$ vanishes everywhere in space is described as **irrotational**. Note that the situation where the fluid flow encounters a solid surface is more complicated because we can no longer draw circular contours around pathlines sufficiently near this object.

If the fluid velocity field is irrotational, then it can be expressed as the gradient of a velocity potential: $\mathbf{v} = \nabla\phi$, which must satisfy Laplace's equation:

$$\nabla^2\phi = 0 \tag{9.7.8}$$

Consequently Euler's equation becomes

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}v^2 + h \right) = 0 \tag{9.7.9}$$

and thus dotting to the left by $\frac{d\mathbf{r}}{ds}$, the tangent vector to an arbitrary curve in the fluid, we find

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}v^2 + h = c(t) \tag{9.7.10}$$

where we can rescale the RHS by redefining the potential. For steady flows the potential

does not vary in time and we get that

$$\frac{1}{2}v^2 + h = c(t) \tag{9.7.11}$$

which is Bernoulli's equation for potential flows. If we further assume constant density and a body force **F** then

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \Phi = c \tag{9.7.12}$$

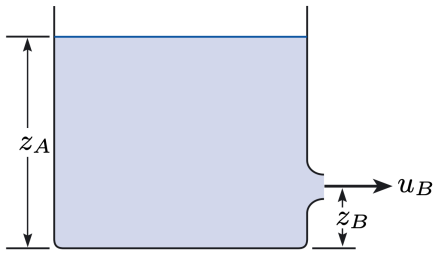
Unlike the more general Bernoulli equation, (9.7.11) applies anywhere in the fluid! The reason this happened was we did not have the pesky $\mathbf{v} \times (\nabla \times \mathbf{v})$ term which would not have vanished when dotted with the tangent vector to any arbitrary curve, but only to streamlines.

To summarise, we have that at constant density, inviscid flow with conservative body force

Type of flow	$b = \frac{1}{2}v^2 + \frac{p}{\rho} + \Phi$
steady, rotational	is constant along a **streamline**
steady, irrotational	is constant along any curve in the fluid
unsteady, irrotational	$= -\frac{\partial \phi}{\partial t} + c(t)$ along any curve in the fluid

9.8 Applications of Bernoulli's equation

Let's take a large tank containing an inviscid liquid at constant density ρ, depth z_A, and with the free surface at rest and at atmospheric pressure p_0. We puncture a small hole in the tank at a height z_B from the bottom, and ask ourselves what the speed of the liquid is as it exits this orifice.



Assuming that the water is constantly added so that the depth is constant at z_A then the flow is steady with the only body force being gravity, which is conservative. For such fluid flows we know that Bernoulli's theorem can be applied (only unsteady rotational flows don't have a Bernoulli-type relation). Let's take point A to lie on the free surface so that it can be connected to point B on the hole via a streamline. We then find that

$$\frac{1}{2}v_B^2 + p_0 + gz_B = p_0 + gz_A \implies v_B = \sqrt{2g(z_A - z_B)} \tag{9.8.1}$$

Let's now assume that the tank is not replenished constantly so that the free surface moves down at a constant speed $\dot{z}_A = v_A$. Since the flow is no longer steady, we cannot expect Bernoulli's equation to apply anymore. However assuming that the free surface is sufficiently large compared to the hole's surface one can still approximately apply Bernoulli. This time we find that

$$\frac{1}{2}v_B^2 + p_0 + gz_B = \frac{1}{2}v_A^2 + p_0 + gz_A \tag{9.8.2}$$

The continuity equation also requires that

$$v_A S_A = v_B S_B \tag{9.8.3}$$

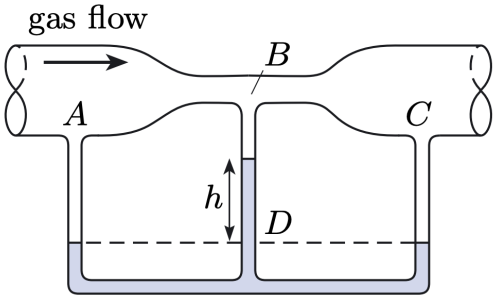
where S_A and S_B are the areas of the free surface and of the hole respectively. Combining the two equations we find that

$$\frac{1}{2}v_B^2\left(1 - \frac{S_B^2}{S_A^2}\right) = g(z_A - z_B) \implies v_B = \sqrt{\frac{2g(z_A - z_B)}{1 - (S_B/S_A)^2}} \tag{9.8.4}$$

Qualitatively we may phrase Bernoulli's equation as

$$\text{high (low) speed implies low (high) pressure} \tag{9.8.5}$$

We can demonstrate this principle quite easily by performing the following experiment. We take a horizontal, uniform circular pipe which contracts for a short length, and is connected via three small vertical tubes, two connecting in the large diameter region and one connecting in the small diameter region. We then let a gas flow through the pipe and place a small amount of fluid in the three small tubes. When the gas passes through the contrac-



tion of the pipe it must accelerate by the continuity equation. This implies that the middle vertical tube will experience a smaller air pressure compared to the other two, causing the fluid to rise. This basic experiment is a miniature model of a paint spray can. As air is pumped through a contracted pipe, connected via a tube to a paint reservoir, the paint will rise up the tube and be carried by the air flow forming droplets.

Let's put these words into equations. Consider a pipe with cross section S_A except for a small region where it contracts to a cross section S_B. As an ideal fluid flows through this pipe the continuity equation imposes that

$$v_A S_B = v_B S_B \tag{9.8.6}$$

where v_A and v_B are the fluid velocities at A and B. Along any streamline between A and B we also find that

$$\frac{1}{2}\rho v_A^2 + p_A = \frac{1}{2}\rho v_B^2 + p_B \tag{9.8.7}$$

so that

$$p_B - p_A = \frac{1}{2}\rho v_A^2\left(1 - \frac{S_A^2}{S_B^2}\right) \tag{9.8.8}$$

This is the pressure difference between the free surface in the tube connecting to A and the free surface in the tube connecting to B. If the fluid rises by a height h then $p_B - p_A = \rho g h$ implying that

$$h = \frac{v_A^2}{2g}\left(1 - \frac{S_A^2}{S_B^2}\right) \tag{9.8.9}$$

is the rise in height of the fluid in the middle tube.

As a final application let's look at the flow of an inviscid fluid over a solid surface. For example, the steady flow of an inviscid constant density liquid past a sphere of radius a is given by

$$\mathbf{v}(r,\theta) = U\cos\theta\left(1 - \frac{a^3}{r^3}\right)\mathbf{r} - U\sin\theta\left(1 + \frac{a^3}{2r^3}\right)\boldsymbol{\theta} \tag{9.8.10}$$

Note that as $r \to \infty$ then $\mathbf{v} \to U(\cos\theta\mathbf{r} - \sin\theta\boldsymbol{\theta})$. Also,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{r} & r\boldsymbol{\theta} & r\sin\theta\boldsymbol{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ U\cos\theta(1-\frac{a^3}{r^3}) & -U\sin\theta(r+\frac{a^3}{2r^2}) & 0 \end{vmatrix} \tag{9.8.11}$$

$$= r\sin\theta\left[-U\sin\theta\left(1 - \frac{a^3}{r^3}\right) + U\sin\theta\left(1 - \frac{a^3}{r^3}\right)\right]\boldsymbol{\phi} = 0 \tag{9.8.12}$$

so the flow is irrotational. In the absence of body forces, and assuming that the pressure at infinity is p_0, then we may apply Bernoulli's equation to the point (r,θ) along any curve

$$\frac{1}{2}(v(a,\theta))^2 + \frac{p(r,\theta)}{\rho} = \frac{1}{2}U^2 + \frac{p_0}{\rho} \tag{9.8.13}$$

Note that

$$(v(a,\theta))^2 = \frac{9}{4}U^2\sin^2\theta \tag{9.8.14}$$

implying that the pressure distribution on the sphere is

$$p(r,\theta) = p_0 + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right) \tag{9.8.15}$$

9.9 Open channel flows

Open channel flows are fluid flows whose cross-section is not completely determined by the shape of the solid boundary confining the liquid (usually a fluid flowing through an open pipe). Open channel flows can get quite complicated, so we will assume that

(i) the liquid is inviscid and has constant density

(ii) the flow is steady

(iii) the channel is straight with a rectangular cross-section of constant width

(iv) the flow is uniform

Suppose that the depth of the liqid is h, and the width of the channel is b. The continuity equation implies that $h_1 b v_1 = h_2 b v_2$ along any point in the channel, and since the width b is taken to be constant we get that the flow rate per unit width

$$Q = hv \text{ is constant} \tag{9.9.1}$$

Lets apply Bernoulli's equation along a streamline on the free-surface of the liquid where the channel is horizontally flat, then:

$$\frac{p_0}{\rho} + \frac{1}{2}v^2 + gh = \text{cnst.} \tag{9.9.2}$$

and since p_0, ρ are both constant, the first being equal to the atmospheric pressure, then we find that

$$E = \frac{v^2}{2g} + h \text{ is constant} \tag{9.9.3}$$

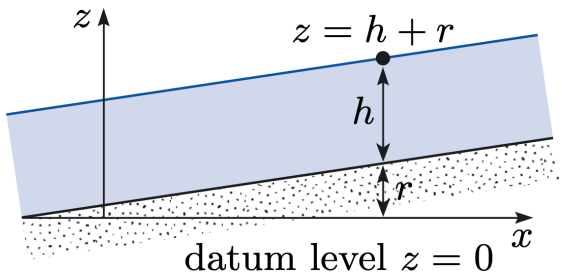
Here E is known as the **specific energy**. Using $Q = hv$ as a constant of motion then we alternatively get

$$E = \frac{Q^2}{2gh} + h \implies h^3 - Eh^2 + \frac{Q^2}{2g} = 0 \tag{9.9.4}$$

This cubic equation has at most three real roots, one of which is always negative and thus unphysical. Therefore for any given E and Q there are either two possible values of the water depth, just one or none. If instead the channel slopes upwards then

$$H = \frac{v^2}{2g} + h + r \tag{9.9.5}$$

where h, r are defined as shown below.



Plots of E as a function of h are known as specific energy curves. We see that usually E takes a minimum value E_{min} at some critical height h_c.

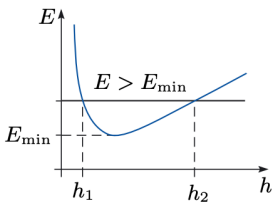


Figure 3.5

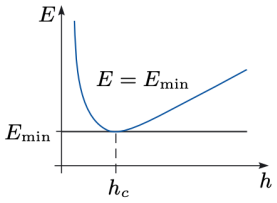
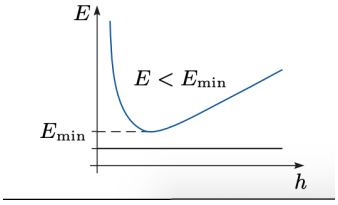


Figure 3.6



If $E > E_{min}$ then there are two physical solutions h_1, h_2 for the water-depth. One, h_2 corresponds to a deep but slow water flow, while the other, h_1, corresponds to a shallow but fast water flow. If instead $E = E_{min}$ there is only one physical water depth, the critical depth h_c with critical speed $v_c = \frac{Q}{h_c}$. Finally if $E < E_{min}$ there is no possible flow.

Finding the critical depth is simple, one need only to minimize E. In the case of a flat channel we find that

$$\frac{dE}{dh} = 1 - \frac{Q^2}{gh_c^3} = 0 \implies h_c = \left(\frac{Q^2}{g}\right)^{1/3} \tag{9.9.6}$$

and therefore

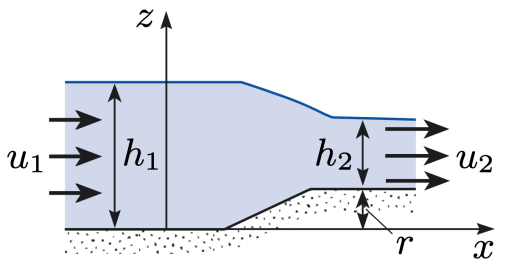
$$v_c = (gQ)^{1/3}, \; E_{min} = \frac{3}{2}\left(\frac{Q^2}{g}\right)^{1/3} \tag{9.9.7}$$

The **Froude number** Fr is defined as

$$Fr = \frac{v}{\sqrt{gh}} = \left(\frac{h_c}{h}\right)^{3/2}\left(\frac{Q^2}{g}\right)^{-1/2}\sqrt{\frac{h^2}{g}}v = \left(\frac{h_c}{h}\right)^{3/2} \tag{9.9.8}$$

so if $Fr < 1$ then we get deep, slow flows, known as **subcritical flows** while if $Fr > 1$ then we get shallow, fast flows, known as **supercritical flows**.

Let's now consider the case where the channel is not horizontal, as shown below



Once again the continuity equation requires

$$u_1 h_1 = u_2 h_2 \tag{9.9.9}$$

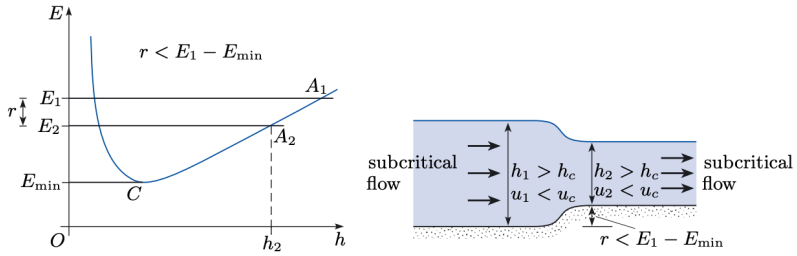
Instead, Euler's equation gives

$$\frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + r \tag{9.9.10}$$

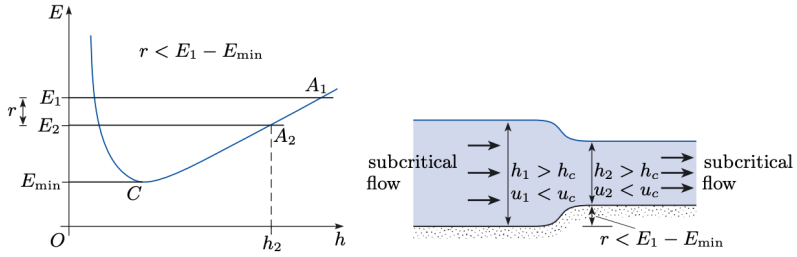
and defining $E = \frac{Q^2}{2gh} + h$ then

$$E_1 = E_2 + r \tag{9.9.11}$$

Let A_1 denote the specific energy E_1 for an upstream subcritical flow. If r is sufficiently small i.e. $E_1 - E_{min} > r$ then the decrease in the specific energy by r yields a new point A_2 whose specific energy satisfies $E_2 > E_{min}$, so that we get a downstream subcritical flow.



If instead $r = E_1 - E_{min}$ then A_2 is on the critical point and the downstream flow will be critical. If r is further increased beyond $E_1 - E_{min}$ then the upstream conditions are not sufficient to sustain a critical flow downstream, thus breaking down the steady flow in the figure above. Because all the fluid cannot go over the rise there will be an increase in the upstream depth which will increase $E_1 - E_{min}$ (pushing A_1 to the right) until steady flow can be re-established. This occurs when $E_1 - E_{min}$ hits the value of r, which will induce a critical downstream flow.



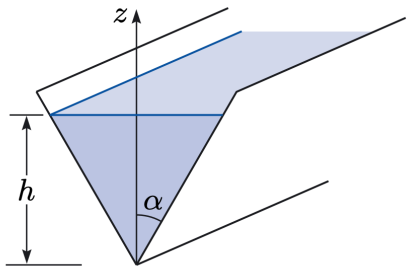
A similar argument can be used for supercitical flow. Note however that if $E < E_{min}$ then it is not possible to have any flow. So if the jump in the channel height is too large then steady flow cannot be established anymore. In the case of subcritical flow this just resulted in a . However, if the flow is supercritical this mechanism cannot work anymore. In such cases the flow suddenly jumps to subcritical (this is known as a **hydraulic jump**).

As an example, consider the steady flow in a horizontal channel with a triangular cross-section. The continuity equation requires

$$v_1 h_1^2 \tan\alpha = v_2 h_2^2 \tan\alpha \implies Q = vh^2 = \text{cnst.} \tag{9.9.12}$$

Therefore, using Bernoulli's equation along a streamline on the fluid's free surface then

$$E = \frac{v^2}{2g} + h = \frac{Q^2}{2gh^4} + h = \text{cnst.} \tag{9.9.13}$$



The critical depth can be found by minimising E

$$1 - \frac{2Q^2}{gh_c^5} = 0 \implies h_c = \left(\frac{2Q^2}{g}\right)^{1/5} \tag{9.9.14}$$

The critical velocity is then

$$v_c = \frac{Q}{h_c^2} = \left(\frac{g^2 Q}{4}\right)^{1/5} \tag{9.9.15}$$

9.10 Vorticity

Let us construct a model for the air flow in a tornado. We make the observations that inside a tornado there is a cylindrical column of air undergoing rapid rotation, while outside this column the air speed decreases dramatically. A suitable velocity vector field for such flows is thus

$$\mathbf{u} = \begin{cases} \Omega_1 r \mathbf{e}_\theta, \ r \leq a \\ \frac{\Omega_2}{r}\mathbf{e}_\theta, \ r > a \end{cases} \tag{9.10.1}$$

where Ω_1 and Ω_2 are the angular velocities inside and outside the tornado respectively. Note that the velocity vector field cannot be discontinuous at $r = a$ in a steady state so

$$\Omega_1 a = \frac{\Omega_2}{a} \implies \Omega_2 = \Omega_1 a^2 \tag{9.10.2}$$

and therefore

$$\mathbf{u} = \begin{cases} \Omega_1 r \mathbf{e}_\theta \\ \frac{\Omega_1 a^2}{r}\mathbf{e}_\theta \end{cases} \tag{9.10.3}$$

Let's now take the limit $A \to 0$ and $\Omega \to \infty$ while maintaining the circulation $\kappa = 2\pi\Omega a^2$ constant. We are therefore making the vortex localised at the origin.In this limit the velocity field reduces to

$$\mathbf{u} = \frac{\kappa}{2\pi r}\mathbf{e}_\theta, \ r \neq 0 \tag{9.10.4}$$

Note that $\nabla \times \mathbf{u} = 0$ for $\neq 0$, implying that the circulation of $\mathbf{u}$ about any path $\mathcal{C}$ not enclosing the origin will vanish. Similarly, the circulation of $\mathbf{u}$ about any path $\mathcal{C}$ enclosing the origin will just be equal to κ. This can be seen by taking two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ and making two small cuts infinitesimally close to each other. The new closed path $S \equiv A_1A_2D_2B_2B_1D_1A_1$

with boundary C does not enclose the line-vortex, so that

$$\oint_S \boldsymbol{\omega} \cdot d\mathbf{l} = 0 \implies \int_C \mathbf{u} \cdot d\mathbf{r} = \int_{C_1} \mathbf{u} \cdot d\mathbf{r} - \int_{C_2} \mathbf{u} \cdot d\mathbf{r} = 0 \tag{9.10.5}$$

since $\boldsymbol{\omega} = 0$ on S. Taking C_2 to be a circle centered at the origin of radius r, so that its circulation is κ, then we find that

$$\int_{C_1} \mathbf{u} \cdot d\mathbf{r} = \kappa \tag{9.10.6}$$

This model of an infinitesimal localised vortex is known as a **line vortex**. The stream function for a line vortex is given by solving

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \text{ and } \frac{\partial \psi}{\partial r} = -\frac{\kappa}{2\pi r} \tag{9.10.7}$$

The first implies that $\psi = c_0 + f(r)$ and while the second implies that $f(r) = -\frac{\kappa}{2\pi}\ln r$, so that

$$\psi(r) = c - \frac{\kappa}{2\pi}\ln r \tag{9.10.8}$$

It will be convenient to consider the streamfunction with $c = \frac{\kappa}{2\pi}\ln a$ so that

$$\psi(r) = -\frac{\kappa}{2\pi}\ln\frac{r}{a} \tag{9.10.9}$$

By the principle of superposition, given two line vortices then the overall streamfunction will be the sum of the individual vortices' streamfunctions.

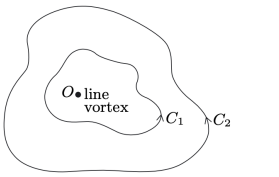
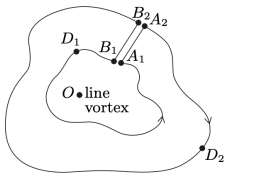
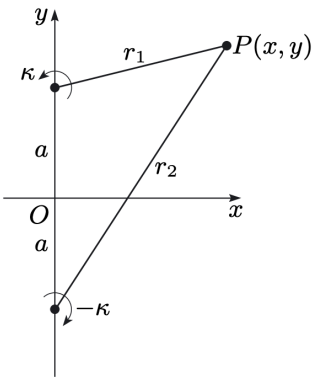


Figure 1.10 Curves C_1, C_2



(a)



(b)

For example, if we have two vortices of circulations $\pm\kappa$ at $(0, \pm a, 0)$, with axis of rotation parallel to the z axis then

$$\psi = -\frac{\kappa}{2\pi}\ln\frac{r_1}{r_2} + c \tag{9.10.10}$$

where r_1 and r_2 are the distances from the top and bottom vortices respectively. To find

the streamlines, let's set $c = 0$ for simplicity, in which case

$$-\frac{\kappa}{2\pi}\ln\frac{r_1}{r_2} = M \implies \frac{r_1^2}{r_2^2}e^{\frac{4\pi M}{\kappa}} \tag{9.10.11}$$

In cartesian coordinates we find that $r_1^2 = x^2 + (y-a)^2$ and $r_2^2 = x^2 + (y+a)^2$, so letting $\alpha = e^{\frac{4\pi M}{\kappa}}$

$$x^2 + (y-a)^2 - \alpha(x^2 + (y+a)^2) = 0 \implies x^2 + \left(y + \frac{M+1}{M-1}a\right)^2 = \frac{4Ma^2}{(M-1)^2} \tag{9.10.12}$$

As long as $M \neq 1$ then the streamlines are functions with center at $\left(0, \frac{M+1}{1-M}a\right)$ and radius $r = |\frac{2\sqrt{M}a}{M-1}|$. If $M = 1$ then $r_1 = r_2$, which is yet another streamline, and corresponds to the x-axis.

Line vortices can also be used to model fluid flows past solid objects. For example, suppose a fluid flows past a rigid cylinder of radius a. The fluid flow can be decomposed into a streaming behaviour described by a stream function and the swirling motion described by a line vortex.

For steady flow of an inviscid, constant density fluid, Bernoulli's equation implies that the pressure at a point B on the cylinder satisfies

$$p + \frac{1}{2}\rho u^2 = p + \frac{1}{2}\rho\left(\frac{\kappa}{2\pi a}\right)^2 = M \tag{9.10.13}$$

where M is a constant. The pressure distribution thus reads

$$p(a,\theta) = M - \frac{\rho\kappa^2}{8\pi^2 a^2} \tag{9.10.14}$$

Since this pressure distribution is constant, the force due to fluid pressure on the cylinder must be zero. Let's now try to add the streaming motion to our study.

The streamlined flow past a cylinder is given by the following streamfunction

$$\psi = Ur\sin\theta - \frac{Ua^2\sin\theta}{r} \tag{9.10.15}$$

and the corresponding velocity vector field is

$$\mathbf{u} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\mathbf{e}_r - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\mathbf{e}_\theta \tag{9.10.16}$$

from which it follows that $\boldsymbol{\omega} = 0$.

$$\oint_C \mathbf{u}\cdot d\mathbf{r} = \int_S \boldsymbol{\omega}\cdot d\mathbf{A} = 0 \tag{9.10.17}$$

Adding the line vortex stream-function we find that

$$\psi = Ur\sin\theta - \frac{Ua^2\sin\theta}{r} + \frac{\kappa}{2\pi}\ln\frac{r}{a} \tag{9.10.18}$$

and thus

$$\mathbf{u} = U\left(1-\frac{a^2}{r^2}\right)\cos\theta\mathbf{e}_r - U\left(1+\frac{a^2}{r^2}\right)\sin\theta\mathbf{e}_\theta - \frac{\kappa}{2\pi r}\mathbf{e}_\theta \implies \oint_C \mathbf{u}\cdot d\mathbf{r} = -\kappa \tag{9.10.19}$$

around any path C enclosing the origin. Now taking B to be the point (a,θ) on the cylinder then we find

$$\mathbf{u}(a,\theta) = -\left(2U\sin\theta + \frac{\kappa}{2\pi a}\right)\mathbf{e}_\theta \tag{9.10.20}$$

so

$$p(a,\theta) + \frac{1}{2}\rho u^2 = p(a,\theta) + \frac{1}{2}\rho\left[4U^2\sin^2\theta + \frac{\kappa^2}{4\pi^2a^2} + \frac{2U\kappa\sin\theta}{\pi a}\right] = M \tag{9.10.21}$$

and thus

$$p(a,\theta) = M - \frac{1}{2}\rho\left[4U^2\sin^2\theta + \frac{\kappa^2}{4\pi^2a^2} + \frac{2U\kappa\sin\theta}{\pi a}\right] \tag{9.10.22}$$

This pressure is not constant and homogeneous over the cylinder's surface. Its upward component will generate a **lift force** while its horizontal component will generate a **drag force**. The lift force per unit length can be computed as

$$f_{\text{lift}} = \int_{-\pi}^{\pi} -p\sin\theta\, a d\theta = \frac{1}{2}\rho a\frac{2U\kappa}{\pi a}\int \sin^2\theta d\theta = \rho\kappa U \tag{9.10.23}$$

since the other terms in p are even in θ. This latter fact is important, because it means that for the lift force to exist we need the term $\frac{2U\kappa\sin\theta}{\pi a}$ which couples the swirling motion with the streamlining behaviour, it cannot exist without both dynamics. Note that the constant M is physically insignificant (as it should be) since it vanishes upon integration. Note that the lift force is proportional to the fluid density, velocity and circulation as one would expect.

Similarly the drag force per unit lengthis given by

$$f_{\text{lift}} = \int_{-\pi}^{\pi} -p\cos\theta\, a d\theta = 0 \tag{9.10.24}$$

9.11 Incompressibility

Another important case where equations of motion are simple enough is that if incompressible fluids i.e. ρ is constant in time and space. The equation of continuity takes the form

$$\nabla\cdot\mathbf{v} = 0 \tag{9.11.1}$$

so the fluid velocity field must be divergenceless. For two-dimensional flow where **v** is constant along one direction then

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{9.11.2}$$

This condition is satisfied by defining a streamfunction ψ such that

$$v_x = \frac{\partial \psi}{\partial y},\ v_y = \frac{\partial \psi}{\partial x} \tag{9.11.3}$$

The streamfunction allows us to compute the shape of streamlines, which we write mathematically as curves $y(x) = x$, by imposing that **v** be tangent to the streamlines:

$$\frac{dy}{dx} = \frac{v_y}{v_x} \implies d\psi = 0 \implies \psi(x, y) = c \tag{9.11.4}$$

Therefore, setting the streamfunction to an arbitrary constant will yield a streamline. The constant is fixed by providing a point on the desired streamline.

The equation of motion also simplifies to

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g} \tag{9.11.5}$$

Notice that $dw = \frac{dp}{\rho} \implies w = \frac{p}{\rho} + c$ and thus Bernoulli's equation becomes

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz = c \tag{9.11.6}$$

Combining irrotational flow with incompressible flow we get Bernoulli's equation over all of space:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} = 0 \tag{9.11.7}$$

If **g** is constant then the maximum pressure $p_{\max}$ occurs at a **stagnation point**, where the fluid velocity vanishes, and takes the form

$$p_{\max} = p_0 + \frac{1}{2}\rho u^2 \tag{9.11.8}$$

where p_0 and **u** are the fluid's pressure and velocity at infinity.

Navier Stokes equation and Viscous fluid flow

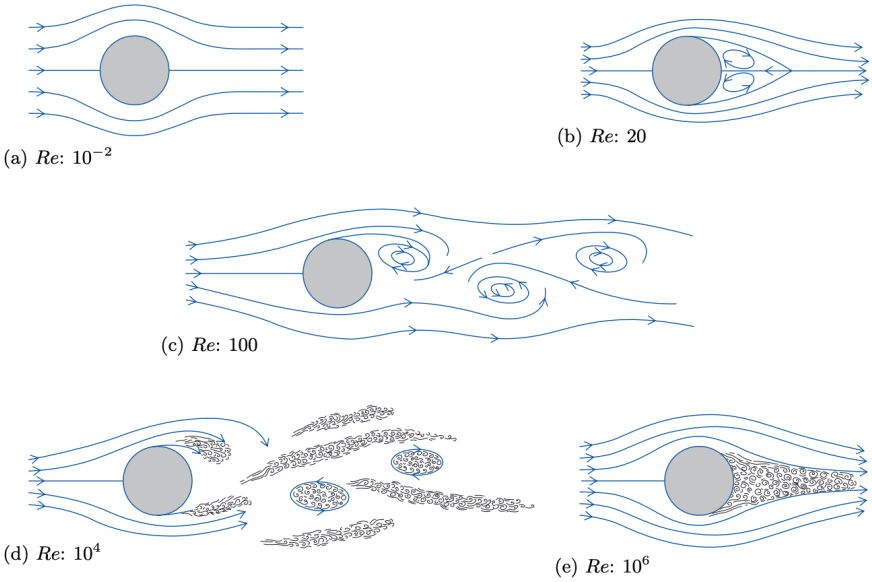
10

10.1 Reynolds number and convection vs diffusion

Suppose we have a fluid of density ρ and coefficient of viscosity μ flowing over a cylinder of radius a, such that at long distances away from the cylinder the fluid speed is U. The **Reynolds number** Re associated to this flow is given by

$$\mathrm{Re} = \frac{\rho U a}{\mu} \tag{10.1.1}$$

Flows for different Reynolds numbers are shown below We see that as we increase the



(a) *Re*: 10^{-2}

(b) *Re*: 20

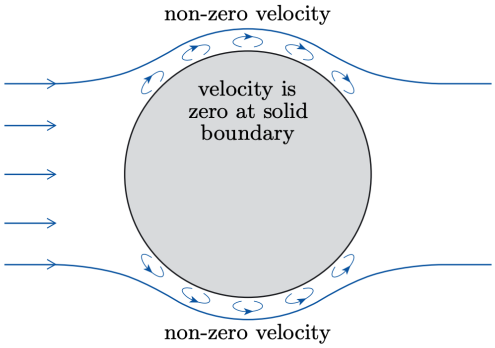
(c) *Re*: 100

(d) *Re*: 10^4

(e) *Re*: 10^6

Reynold's number the flow becomes increasingly more complex and turbulent. One can understand these different behaviours as competitions between convection and diffusion transport processes of vorticity in the fluid, the former characterised by fluid elements moving and transporting the vorticity with them, while the latter is characterised by the viscosity (internal friction) between the fluid particles breaking away the vorticity of the fluid.

To understand how vortices can arise in the first place, consider the following flow past a cylinder.

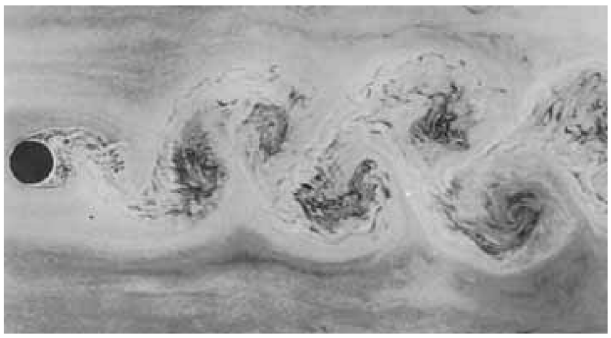


For a viscous fluid, the adhesive forces between the fluid molecules and the solid boundary particles are much stronger than the cohesive forces. As such, we can formulate a no-slip condition where the fluid velocity at the solid boundary must vanish. Away from the surface however the fluid velocity is non-zero, and this steep velocity gradient produces vortices in the neighbourhood of the solid boundary. Once a vortex has formed, it can be transported convectively by the fluid particles, or diffusively due to internal friction mechanism, in which case it is dissipated away.

Suppose we hold μ, a, ρ constant and vary U, the main flow speed. At small U, there is little difference the velocity gradient near the solid boundary is shallow so few vortices will be generated. Furthermore, dissipation will be the principal transport process, far more prevalent than convection as each fluid element will not travel far. Therefore the vortices will generally have a very short lifetime. As a result the flow will look steady and orderly.

Let's now increase U. Now the velocity gradient is larger so vortices can start forming near the solid boundary. Also, convection starts to become more prevalent and diffusion will have less time to dissipate vorticity. This will produce a boundary layer where vortices are confined to, outside of which diffusion is strong enough to dissipate them away.

A further increase in U will enhance convenction and reduce diffusion, leading to instabilities in the bound vortices which will start to grow in side and interact with each other. This will result in one of the vortices breaking off, and forming what's known as a **Karman vortex streets**:



To put this analysis in quantitative terms let's use dimensional analysis to find relevant time-scales for vorticity according in the regimes where convection or diffusion dominate.

Let's set

$$t = \mu^{\alpha}\rho^{\beta}U^{\gamma}a^{\delta} \tag{10.1.2}$$

We have three different units, M, L, T and four unknowns, implying that several different combinations can be found. Noting that

$$[\mu] = M \cdot L^{-1} \cdot T^{-1},\ [\rho] = M \cdot L^{-3},\ [U] = L \cdot T^{-1},\ [a] = L \tag{10.1.3}$$

we find that

$$\begin{cases} \alpha + \beta = 0 \\ -\alpha - 3\beta + \gamma + \delta = 0 \\ -\alpha - \gamma = 1 \end{cases} \implies \begin{cases} \alpha = -\beta \\ \gamma + \delta = 2\beta \\ \beta = 1 + \gamma \end{cases} \implies \begin{cases} \alpha = -\beta \\ \delta = \beta + 1 \\ \gamma = \beta - 1 \end{cases} \tag{10.1.4}$$

Setting $\beta = 1$ then $\alpha = -1,\ \delta = 2,\ \gamma = 0$ so we get $t = \frac{\rho a^2}{\mu}$. If $\beta = 0$ then $\alpha = 0, \delta = 1, \gamma = -1$ so we get $t = \frac{a}{U}$. Finally if $\beta = -1$ then $\alpha = 1, \delta = 0, \gamma = -2$ so that $t = \frac{\mu}{\rho U^2}$. To summarise, we have found that the following time scales may be relevant

$$t = \frac{a^2\rho}{\mu},\ \frac{a}{U},\ \frac{\mu}{\rho U^2} \tag{10.1.5}$$

To determine which one of these is characteristic of convenction, note that the time it takes for a fluid particle with speed U to cross the cylinder is $\frac{2a}{U}$ so $t_c = \frac{a}{U}$ is the convective time scale. For diffusion, we should expect the corresponding time scale to decrease with viscosity, since the larger μ is the less it will take to dissipate a vortex. Consequently we find that $t_d = \frac{\mu}{\rho U^2}$ is the diffusive time scale. Note that

$$\text{Re} = \frac{t_c}{t_d} \tag{10.1.6}$$

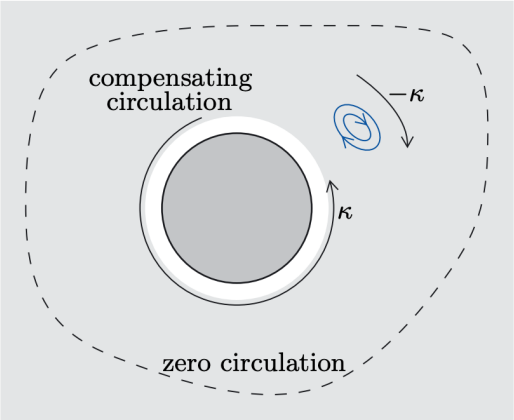
so the Reynolds number is a measure of the competition between convection processes and diffusion processes. We now see that if we want the vorticity to remain confined near the cylinder then we need the time it takes for vorticity to be transported convectively to be much smaller than the time it takes for it to be dissipated. Therefore

$$t_c \gg t_d \implies \frac{a}{U} \gg \frac{\mu}{\rho U^2} \implies \text{Re} \gg 1 \tag{10.1.7}$$

just as we explained previously in words.

Lets now look at vortex shedding in more detail. Suppose a vortex has been shed from a fluid flow, and is carried by convective transport. Let's also take a large curve surrounding both the cylinder and the shed vortex. By Kelvin's theorem (applicable at the initial stages), the circulation around this curve must remain constant. Since the shed vortex has non-zero circulation once it reaches the large curve, the flow around the cylinder must compensate with an opposite vortex. Since fluid mass will be moved upwards, this will generate a downward force on the cylinder. This will in turn shed a vortex which swirls

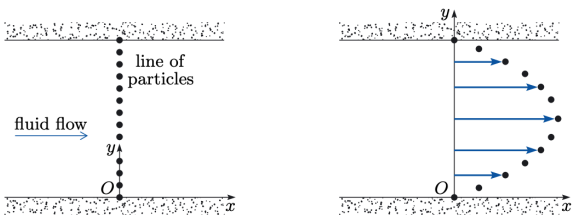
in the opposite direction to the first vortex, thus repeating the argument. Throughout this



process the cylinder experiences periodic forces moving it sideways. If the frequency of this force matches the natural frequency of the cylinder then we achieve resonance. This can lead to disastrous consequences, such as the collapse of the Tacoma bridge.

10.2 Newton's model of viscosity

Consider a fluid flowing through a pipe steadily from left to right. At time $t = 0$ we place a line of massless particles to keep track of the fluid flow, and observe that the fluid speed seems largest near the middle of the pipe, and smallest near its solid boundary. This is unccounted by the inviscid fluid theory where the particles should all move together and maintain their shape. The explanation for this phenomenon lies in the fact that fluids are



viscous and there are internal friction forces acting within the fluid. We note that the particles form a parabolic shape and do not move at all at the edges of the pipe. Such pictures of the velocity of the fluid against transverse position is known as a **velocity profile**. The velocity profile is vertical in an inviscid fluid and parabolic in the viscous case.

Consider a viscous, incompressible fluid between two plates, the upper plate moving with speed U relative the the lower plate which is fixed. The continuity equation implies that the fluid velocity obey $u = u_x(z)\mathbf{x}$.

Now consider two points $P = (x, z)$ and $Q(x, z + \delta z)$, and suppose that after time δt the fluid particles originally at these points have moved to P' and Q'. We see that for small $\delta\alpha$

(so sufficiently small δt and δz) then

$$\delta\alpha \approx \tan\alpha = \frac{AQ'}{PQ} = \frac{(u+\delta u)\delta t - u\delta t}{\delta z} == \frac{\delta u \cdot \delta t}{\delta z} \tag{10.2.1}$$

Consequently the rate of deformation, $\frac{d\alpha}{dt}$ is given by

$$\frac{d\alpha}{dt} = \frac{du}{dz} \tag{10.2.2}$$

Finally,assuming the shear stress σ is proportional to the rate of deformation, then

$$\sigma = \mu\frac{d\alpha}{dt} = \frac{du}{dz} \tag{10.2.3}$$

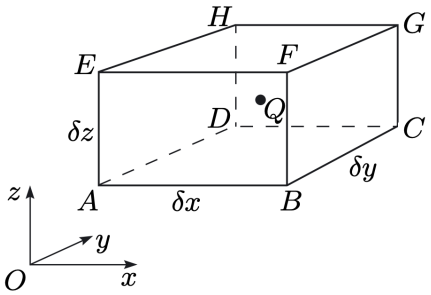
For unsteady flows it can be shown similarly that

$$\sigma = \mu\frac{\partial u}{\partial z} \tag{10.2.4}$$

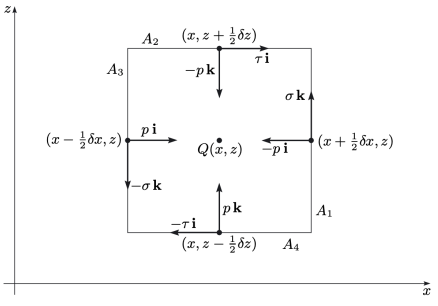
This is **Newton's model of viscosity** and fluids satisfying this law are known as **Newtonian fluids**. If a Newtonian fluid with velocity $\mathbf{u} = u(x)\mathbf{z}$ is placed between two parallel plates in the yz-plane, with the left plate held fixed and the right plate moving upwards with constant speed U, then the fluid will start moving with the plate due to shear stress. The shear stress on a surface parallel to the yz-plane at $x = x_0$ is $\tau = \mu\frac{\partial u}{\partial x}\Big|_{x_0}$. The force acting to the right of an area A of the plane is $\tau A\mathbf{z}$, while the force acting to the left of the plane is $-\tau A\mathbf{z}$.

Now consider a Newtonian fluid with constant density and viscosity flowing between two infinite horizontal plates a distance h apart, lying in the xy-plane. We assume that the velocity and pressure are independent of y and the flow is in the x-direction only, and dependent on z, t as required by the continuity equation. We can therefore write the velocity field as $\mathbf{u} = y_1(z,t)\mathbf{x}$. We will also ignore body forces for now and add them at the end.

Let's consider an infinitesimal fluid element centered at Q and its cross-section through Q The surface forces acting on each face of the element are either pressure forces, pointing



(a)



(b)

radially inwards, or shear stresses, pointing tangentially. The surface force acting on each face of the block will be

$$\mathbf{F}_1 = [-p(x+\delta x/2,z)\hat{\mathbf{e}}_x + \sigma(x+\delta x/2,z)\hat{\mathbf{e}}_z]\delta y\delta z \tag{10.2.5}$$
$$\mathbf{F}_2 = [\tau(x,z+\delta z/2)\hat{\mathbf{e}}_x - p(x,z+\delta z/2)\hat{\mathbf{e}}_z]\delta x\delta y \tag{10.2.6}$$
$$\mathbf{F}_3 = [p(x-\delta x/2,z)\hat{\mathbf{e}}_x - \sigma(x-\delta x/2,z)\hat{\mathbf{e}}_z]\delta y\delta z \tag{10.2.7}$$
$$\mathbf{F}_4 = [-\tau(x,z-\delta z/2)\hat{\mathbf{e}}_x + p(x,z-\delta z/2)\hat{\mathbf{e}}_z]\delta x\delta y \tag{10.2.8}$$

Newton's second law applied to the infinitesimal fluid element then gives

$$\rho\delta x\delta y\delta z\frac{du_x}{dt}\hat{\mathbf{e}}_x = \mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\mathbf{F}_4 \tag{10.2.9}$$

Resolving this equation into components, we find that

$$\rho\frac{du_x}{dt} = -\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial z} \tag{10.2.10}$$
$$0 = \frac{\partial \sigma}{\partial x}-\frac{\partial p}{\partial z} \tag{10.2.11}$$

This describes the translational motion of the fluid, but the rotational degrees of freedom have not been taken into account. The torques on each face are given by the shear stresses only, they are given by

$$\boldsymbol{\tau}_1 = -\frac{1}{2}\sigma(x+\delta x/2,z)\delta x\delta y\delta z\,\hat{\mathbf{e}}_y \tag{10.2.12}$$
$$\boldsymbol{\tau}_2 = \frac{1}{2}\sigma(x,z+\delta z/2)\delta x\delta y\delta z\,\hat{\mathbf{e}}_y \tag{10.2.13}$$
$$\boldsymbol{\tau}_3 = -\frac{1}{2}\sigma(x-\delta x/2,z)\delta x\delta y\delta z\,\hat{\mathbf{e}}_y \tag{10.2.14}$$
$$\boldsymbol{\tau}_4 = \frac{1}{2}\sigma(x,z+\delta z/2)\delta x\delta y\delta z\,\hat{\mathbf{e}}_y \tag{10.2.15}$$

Consequently letting the fluid element have moment of inertia $I = \frac{1}{12}\rho\delta x\delta y\delta z[(\delta x)^2+(\delta z)^2]$ then

$$\frac{1}{12}\rho[(\delta x)^2+(\delta z)^2]\dot{\omega} = -\frac{1}{2}(\sigma(x+\delta x/2)+\sigma(x-\delta x/2))+\frac{1}{2}(\tau(x+\delta x/2)+\tau(x-\delta x/2)) \tag{10.2.16}$$

Taking the limit as $\delta x,\delta z\to 0$ we finally arrive at $\delta(x,z)=\tau(x,z)$. Therefore

$$\frac{\partial\sigma}{\partial x} = \frac{\partial\tau}{\partial x} = \mu\frac{\partial^2 u_x}{\partial z\partial x} = 0 \tag{10.2.17}$$

since the speed u_x only depends on z. It then follows that

$$\frac{\partial p}{\partial z} = 0 \tag{10.2.18}$$

so pressure only varies along x. To conclude we find that

$$\rho\frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + \rho F_x + \mu^2\frac{\partial^2 u_x}{\partial z^2} \tag{10.2.19}$$

where we included a body-force per unit mass F_1 acting along $\hat{\mathbf{e}}_x$.

If we consider a steady flow with a constant pressure gradient $\frac{\partial p}{\partial x} = -C$, then we see that

$$0 = C + \mu^2\frac{\partial^2 u_x}{\partial z^2} \implies u_x(z) = -\frac{C}{2\mu}z^2 + Az + B \tag{10.2.20}$$

10.3 The Navier-Stokes equation

One can easily extend the discussion of the last section and obtain the **Navier-Stokes equations**

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\mathbf{F} + \mu\nabla^2\mathbf{u} \tag{10.3.1}$$

or alternatively

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times\boldsymbol{\omega}\right) = -\nabla p + \rho\mathbf{F} - \mu\nabla\times\boldsymbol{\omega} \tag{10.3.2}$$

The Navier-Stokes equations alone are not sufficient to describe mathematically any Newtonian fluid. Indeed one must also provide suitable boundary conditions. For example, if the fluid is in contact with a solid boundary moving with speed $\mathbf{U}$ and with unit normal $\mathbf{n}$ then we must require

$$\mathbf{u}\cdot\mathbf{n}\big|_{\text{boundary}} = \mathbf{U}\cdot\mathbf{n} \tag{10.3.3}$$

or else the fluid would pass through the solid boundary. We also have another empirically verified condition, known as the **no-slip condition**, which states that a viscous fluid has no tangential velocity relative to a solid boundary next to it. Therefore

$$\mathbf{u}\times\mathbf{n}\big|_{\text{boundary}} = \mathbf{U}\times\mathbf{n} \tag{10.3.4}$$

These two conditions combined together imply that

$$\mathbf{u}\big|_{\text{boundary}} = \mathbf{U} \tag{10.3.5}$$

For example, consider a viscous fluid flowing in a stationary cylindrical pipe of radius a. We find that

$$\mathbf{u}(a,\theta,\phi) = 0 \forall\theta\in[0,\pi),\ \phi\in[0,2\pi) \tag{10.3.6}$$

There is one exception to these boundary conditions. Consider a fluid-fluid interface with the upper fluid stationary relative the the fluid below. The stress-free boundary condition implies that the shear stress at the interface be zero so that

$$\frac{\partial u_x}{\partial z} = 0 \text{ at interface} \tag{10.3.7}$$

Fluid motion in a pipe

Consider a fluid flowing steadily through a pipe with circular cross-section and radius a. Ignoring the influence of gravity, the azimuthal symmetry of the configuration suggests that there should be no ϕ-dependence. Since the pipe is stationary, we find that

$$u_r = u_\varphi = u_z = 0 \; r = a \tag{10.3.8}$$

There also can't be a radial velocity component u_r since the fluid is confined. Consequently the continuity equation yields

$$\frac{\partial u_z}{\partial z} = 0 \implies u_z = u_z(r) \tag{10.3.9}$$

Now the Navier-Stokes equation in cylindrical coordinates gives

$$-\frac{\partial p}{\partial r} + \rho F_r = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho F_\theta = 0 \tag{10.3.10}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho F_Z + \mu\nabla^2 u_z \tag{10.3.11}$$

which reduces to

$$-\frac{\partial p}{\partial r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \tag{10.3.12}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr}\right) \tag{10.3.13}$$

assuming body forces (such as gravity) are negligible. The first two equations imply that $p = p(z)$, and since u_z depends on r only we find that the two terms in the third equation must each be constants. Hence $p(z) = p_0 - Cz$ and

$$\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} + \frac{C}{\mu} = 0 \tag{10.3.14}$$

This is just a Cauchy-Euler equation [1] with general solution

$$u_z(r) = -\frac{Cr^2}{4\mu} + A\ln r + B \tag{10.3.15}$$

Since u_z must be bounded at defined at $r = 0$, we have that $A = 0$. Moreover, $u_z(a) = 0$ implies that $B = \frac{Ca^2}{4\mu}$ and so

$$u_(r) = \frac{C}{4\mu}(a^2 - r^2) \tag{10.3.16}$$

where C is given by the pressure variation in the pipe that is driving the fluid. To see how well this fares with experimental results, let's calculate the volume flow rate (velocity flux)

[1] see Mathematical methods volume

through the pipe's cross-section

$$Q = \int_0^a \int_{-\pi}^{\pi} u_z r \, dr \, d\theta = \frac{\pi C a^4}{8\mu} \tag{10.3.17}$$

Thus the flow rate is proportional to the pressure drop along the pipe and as the fourth power of its radius. This matches the experimental observations made by Hagen and Poiseuille.

10.4 Approximate Navier-Stokes

Fluid injection problem

Consider a viscous fluid of constant density ρ and viscosity μ flowing through a channel of width h formed by two parallel porous plates. The fluid flows due to a constant pressure gradient C, and is also injected through the bottom plate with constant speed V and sucked towards the top plate with the same speed.

We assume that the flow is steady since all external parameters causing the fluid flow are constant. Also, we assume the plates are sufficiently large for edge effects to be negligible, making the velocity field constant at different points along the plates. Finally, we also ignore body forces. We set our axes so that the flow is along z and the plates are normal to the z-axis. Consequently $\mathbf{u} = u_x\mathbf{x} + u_z\mathbf{z}$ and the boundary conditions are

$$u(z = 0) = u(z = h) = V\mathbf{z} \tag{10.4.1}$$

Since the fluid is incompressible we find that

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \tag{10.4.2}$$

Since the fluid flow is the same along the plates, the first derivative must vanish, implying that u_z is independent of both x, z, and thus must be constant $u_z = V$.

The Navier-Stokes equation along **x** is

$$\rho\left(u_x\frac{\partial u_x}{\partial x} + u_z\frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \tag{10.4.3}$$

which simplifies to

$$\rho V\frac{du_x}{dz} - \mu\frac{d^2u_x}{dz^2} = -\frac{\partial p}{\partial x} \tag{10.4.4}$$

The Navier-Stokes equation along **z** and **y** is

$$-\frac{\partial p}{\partial y} = -\frac{\partial p}{\partial z} = 0 \implies p = p(x) \tag{10.4.5}$$

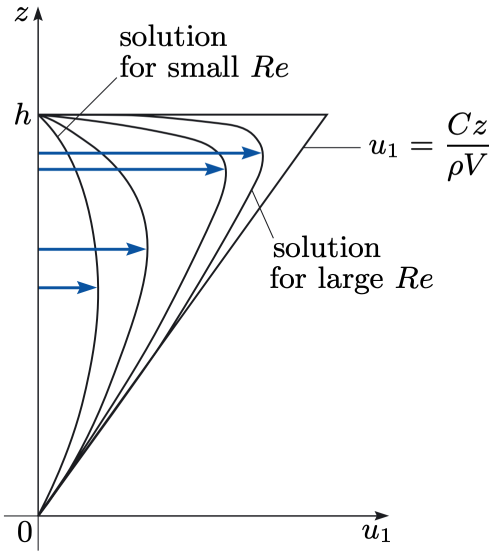
Therefore, we find that since $\nabla p = -C$ then

$$\rho V\frac{du_x}{dz} - \mu\frac{d^2u_x}{dz^2} = C \tag{10.4.6}$$

This is a second order constant coefficient ODE, which together with the no slip condition $u_x(z=0) = u_x(z=h) = 0$ can be solved to give

$$u_x = \frac{C}{\rho V}\left(z - h\frac{e^{\mathrm{Re}z/h} - 1}{e^{\mathrm{Re}} - 1}\right),\ \mathrm{Re} = \frac{\rho V h}{\mu} \tag{10.4.7}$$

The solutions for different Reynolds numbers are shown below. We see that as we increase Re the maximum velocity $u_{\max}$ is achieved at larger and larger z-values, causing the drop of $u_x(z)$ to 0 at $z = h$ to get steeper and steeper. The solution far from the top plate also starts looking more linear, as one would expect from an inviscid model.



Indeed in the large Reynolds number limit (that is in the inviscid limit) then

$$\frac{e^{\mathrm{Re}z/h} - 1}{e^{\mathrm{Re}} - 1} = e^{-\mathrm{Re}}\frac{e^{\mathrm{Re}z/h} - 1}{1 - e^{-\mathrm{Re}}} = \frac{e^{-\mathrm{Re}(1-z/h)} - e^{\mathrm{Re}}}{1 - e^{-\mathrm{Re}}} \approx e^{-\mathrm{Re}(1-z/h)} \tag{10.4.8}$$

so we get that

$$u_x \approx \frac{C}{\rho V}(z - he^{-\mathrm{Re}(1-z/h)}) \tag{10.4.9}$$

Thus when $z \ll h$ the velocity component $u_x = \frac{C}{\rho V}z$ varies linearly in z, and is the solution to the inviscid Navier-Stokes equation:

$$\rho V\frac{du_z}{dz} = C,\ u_x(0) = u_x(h) = 0 \tag{10.4.10}$$

There's a problem here, the ODE is first order and yet we have two conditions to be satisfied. Note also that the general solution is

$$u_x(z) = \frac{C}{\rho V}z + A \tag{10.4.11}$$

so if $u_x(0) = 0$ then $A = 0$ in which case $u_x(h) \neq 0$. Similarly if $u_x(h) = 0$ then $A = -\frac{Ch}{\rho V}$ in

which case $u_x(0) \neq 0$. Therefore the no-slip conditions cannot be simultaneously satisfied in the inviscid model. Indeed, for a low-viscosity fluid the problem cannot be solved by just setting $\mu = 0$. This is because in the neighborhood of the plate at $z = h$, the second order derivative $\frac{d^2u_x}{dz^2}$ is very large and so even if μ is small the resulting term $\mu\frac{d^2u_x}{dz^2}$ still isn't negligible. This narrow region where $\mu\frac{d^2u_x}{dz^2}$ is sufficiently large is known as a **boundary layer**.

Dimensionless Navier-Stokes equation

We saw the significance of the Reynolds number in the fluid injection problem. Since in most cases the Navier-Stokes equations cannot be solved exactly, obtaining a large Reyolds number approximate Navier-Stokes equation can prove very useful.

The ODE for the fluid injection problem was

$$\rho V\frac{du_x}{dz} - \mu\frac{d^2u_x}{dz^2} = C \tag{10.4.12}$$

We choose the following characteristic variables

$$z^* = \frac{z}{h} \text{ and } u_x^* = \frac{u_x}{V} \tag{10.4.13}$$

and get that

$$\frac{\rho V^2}{h}\frac{du_x^*}{dz^*} - \frac{\mu V}{h^2}\frac{d^2u_1^*}{dz^{*2}} = C \tag{10.4.14}$$

or equivalently

$$\frac{du_x^*}{dz^*} = C^* + \frac{1}{\text{Re}}\frac{d^2u_1^*}{dz^{*2}} \tag{10.4.15}$$

where $C^* = \frac{hC}{\rho V^2}$.

Flow past an object

We now consider the flow past an object such as a sphere with a characteristic length a, such that far upstream the fluid has uniform speed U. We can define the following dimensionless variables

$$u_i^* = \frac{u_i}{U} \text{ and } x_i^* = \frac{x_i}{a} \tag{10.4.16}$$

as well as $t^* = t/T$ and $p^* = p/P$ where T, P are undetermined for now. Then the Navier-Stokes equation along **x**

$$\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \tag{10.4.17}$$

becomes

$$\frac{a}{UT}\frac{\partial u_x^*}{\partial t^*} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z} = -\frac{P}{\rho U^2}\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}}\left(\frac{\partial^2 u_x^*}{\partial x^{*2}} + \frac{\partial^2 u_x^*}{\partial y^{*2}} + \frac{\partial^2 u_x^*}{\partial z^{*2}}\right) \tag{10.4.18}$$

It is clear that if we choose $T = \frac{U}{a}$ and $P = \rho U^2$ then we find that

$$\frac{\partial u_x^*}{\partial t^*} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_x^*}{\partial x^{*2}} + \frac{\partial^2 u_x^*}{\partial y^{*2}} + \frac{\partial^2 u_x^*}{\partial z^{*2}}\right) \tag{10.4.19}$$

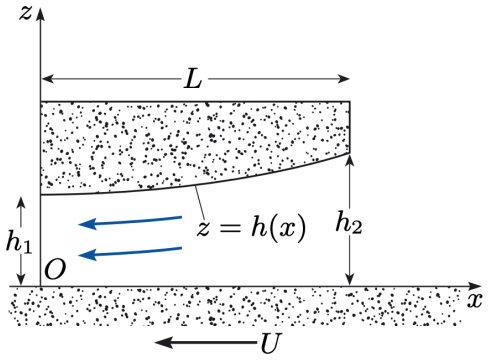
Just like in the fluid injection problem, if we are in the high Reynolds number regime and far away from the object then one can ignore the Laplacian term and find a solution to

$$\frac{\partial u_x^*}{\partial t^*} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{\partial p^*}{\partial x^*} \tag{10.4.20}$$

Near the object however (within the boundary layer) the Laplacian term becomes too large to be ommitted and a full solution must be found. This must then be stitched together with the solution far from the object.

10.5 Dominant viscosity flow

Lubricants work thanks to viscosity. One important application of flows with dominant viscosity is in slider bearings, where lubricants are important in reducing friction between mechanical components.



We assume that the lubricant is a Newtonian viscous fluid with constant density and viscosity. We also assume that the bearing is flat and that the fluid flows in the xz-plane with no y-dependence and no velocity y-component u_y. We also consider the steady state flow. We set our axes so that the slider of length l has a profile defined by $z = h(x)$ and where the bearing guide moves with constant speed U.

The continuity equation is

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{10.5.1}$$

Instead the Navier-Stokes equations are

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_z \frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \tag{10.5.2}$$

$$\rho\left(u_x \frac{\partial u_z}{\partial x} + u_z \frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2}\right) \tag{10.5.3}$$

with boundary conditions

$$u_x(x,0) = -U,\ u_z(x,0) = u_z(x,h(x)) = u_x(x,h(x)) = 0 \tag{10.5.4}$$

Moreover, ignoring edge effects then we also have that

$$p(0,z) = p(L,z) = p_0 \tag{10.5.5}$$

To reduce the Navier-Stokes equation into a dimensionless form, we introduce the following variables

$$x^* = \frac{x}{L},\ z^* = \frac{z}{h_1},\ u_x^* = \frac{u_x}{U},\ u_z?* = \frac{u_z L}{U h_1},\ p^* = \frac{p}{P} \tag{10.5.6}$$

for some characteristic pressure P. We then obtain

$$\frac{h_1^2}{L^2}\left(u_x^*\frac{\partial u_x^*}{\partial x^*} + u_z^*\frac{\partial u_x^*}{\partial z^*}\right) = -\frac{h_1^2}{L^2}\frac{P}{\rho U^2}\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho L U}\left(\frac{h_1^2}{L^2}\frac{\partial^2 u_x^*}{\partial x^{*2}} + \frac{\partial^2 u_x^*}{\partial z^{*2}}\right) \tag{10.5.7}$$

$$\frac{h_1^2}{L^2}\left(u_x^*\frac{\partial u_z^*}{\partial x^*} + u_z^*\frac{\partial u_z^*}{\partial z^*}\right) = -\frac{P}{\rho U^2}\frac{\partial p^*}{\partial z^*} + \frac{\mu}{\rho L U}\left(\frac{h_1^2}{L^2}\frac{\partial^2 u_z^*}{\partial x^{*2}} + \frac{\partial^2 u_z^*}{\partial z^{*2}}\right) \tag{10.5.8}$$

Assuming $\frac{h_1}{L}$ is small and recognising that $\frac{\mu}{\rho L U} = \frac{1}{\text{Re}}$ then we find that the approximate Navier-Stokes equations are

$$0 = -\frac{h_1^2}{L^2}\frac{P}{\rho U^2}\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}}\frac{\partial^2 u_x^*}{\partial z^{*2}} \tag{10.5.9}$$

$$0 = -\frac{P}{\rho U^2}\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}}\frac{\partial^2 u_z^*}{\partial z^{*2}} \tag{10.5.10}$$

To determine what P is, we could make it equal to ρU^2 just as in the case of a flow past an object. However, in our case we have two characteristic length scales so we should not make the characteristic pressure depend on U only (which is characteristic along the x-direction). Any expression of the form $P = \rho U^2 L^\alpha / h_1^\alpha$ will also do the job. If $\alpha > 2$ then the pressure gradient term will obviously dominate since $h_1 \ll L$. One would then find that the pressure gradient is constant $\frac{\partial p}{\partial x} = 0$. Together with $p(0,z) = p(L,z) = p_0$ this implies that a constant pressure is predicted, which does not match observations. Indeed it is the pressure generated by the flow that makes lubricants useful. If $\alpha < 2$ then the pressure gradient is negligible implying that $\frac{\partial^2 u_x}{\partial z^2} = 0$. This is unfortunately a violation of conservation of mass, as we shall see soon. We are left with $\alpha = 2$ and therefore $P = \frac{\rho U^2 L^2}{h_1^2}$.

We are left with

$$0 = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}}\frac{\partial^2 u_x^*}{\partial z^{*2}} \tag{10.5.11}$$

$$0 = -\frac{L^2}{h_1^2}\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}}\frac{\partial^2 u_z^*}{\partial z^{*2}} \tag{10.5.12}$$

Since $L \gg h_1$, the second equation reduces to $\frac{\partial p}{\partial z}$ and therefore $p = p(x)$. The first equation

now looks like

$$\frac{\partial u_x^2}{\partial z^2} = \frac{1}{\mu}\frac{dp}{dx} \tag{10.5.13}$$

where we reverted back to dimensional variables. Since the LHS is dependent on x and z while the RHS depends on x-only we find that

$$u_x(x,z) = \frac{1}{2\mu}\frac{dp}{dx}z^2 + f(x)z + g(x) \tag{10.5.14}$$

Using the boundary conditions we get $g(x) = -U$ and

$$f(x) = \frac{1}{h}\left(U - \frac{h^2}{2\mu}\frac{dp}{dx}\right) \tag{10.5.15}$$

and so

$$u_x(x,z) = -U\left(1 - \frac{z}{h}\right) - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{z}{h}\left(1 - \frac{z}{h}\right) \tag{10.5.16}$$

This is still not very useful as we don't know what $p(x)$ is. To determine it we can calculate the volume flow rate which, by the law of conservation of mass, must be constant. We get that taking a cross-section at some x with $h = h(x)$ then

$$Q = \int_0^h u_x\, dz = \int_0^h \left[-U\left(1 - \frac{z}{h}\right) - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{z}{h}\left(1 - \frac{z}{h}\right)\right] dz = -\frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx} \tag{10.5.17}$$

and thus

$$\frac{dp}{dx} = -\frac{12\mu}{h(x)^3}\left(\frac{1}{2}Uh(x) + Q\right) \tag{10.5.18}$$

We can integrate this with respect to x from 0 to L and note that Q must be constant to find that

$$p(L) - p(0) = 0 = -12\mu\left(\frac{1}{2}U\int_0^L \frac{dx}{(h(x))^2} + Q\int_0^L \frac{dx}{(h(x))^3}\right) \tag{10.5.19}$$

and hence

$$Q = -\frac{1}{2}U\frac{\int_0^L \frac{dx}{(h(x))^2}}{\int_0^L \frac{dx}{(h(x))^3}} \tag{10.5.20}$$

The volume flow rate can be completely determined by the shape of the bearing gap so it can be assumed to be known. Then the pressure can be calculated by solving

$$\frac{dp}{dx} = -6\mu\left(\frac{U}{h^2} + \frac{2Q}{h^3}\right) \tag{10.5.21}$$

which yields

$$p(x) = -6\mu\frac{L}{h_1 - h_2}\left(\frac{U}{h(x)} + \frac{Q}{(h(x))^2}\right) + c \tag{10.5.22}$$

If we take $z = h(x)$ to be a straight line then $h(x) = h_1 + \frac{h_2 - h_1}{L}x$ and hence

$$\int_0^L \frac{dx}{(h(x))^2} = \int_0^L \frac{dx}{(h_1 + x(h_2 - h_1)/L)^2} = \frac{L}{h_1 h_2} \tag{10.5.23}$$

and

$$\int_0^L \frac{dx}{(h(x))^3} = \int_0^L \frac{dx}{(h_1 + x(h_2 - h_1)/L)^3} = \frac{L(h_1 + h_2)}{2h_1^2 h_2^2} \tag{10.5.24}$$

implying that

$$Q = -\frac{1}{U}\frac{L}{h_1 h_2}\frac{2h_1^2 h_2^2}{L(h_1 + h_2)} = -\frac{h_1 h_2}{h_1 + h_2}U \tag{10.5.25}$$

Substituting this into our expression for $p(x)$ one finds that

$$p(x) = -6\mu UL\left(\frac{-h(x)(h_1 + h_2) + h_1 h_2}{(h(x))^2(h_2^2 - h_1^2)}\right) + C \tag{10.5.26}$$

Since $p(0, h_1) = p_0$ we find that

$$p_0 = \frac{6\mu UL}{h_2^2 - h_1^2} + C \tag{10.5.27}$$

finally giving an expression for the pressure

$$p(x) = p_0 - 6\mu UL\frac{(h(x) - h_1)(h(x) - h_2)}{(h_2^2 - h_1^2)(h(x))^2} \tag{10.5.28}$$

The total load per unit length (along z) that the bearing can carry is given by

$$\int_0^L (p - p_0)\, dx = \frac{6\mu UL^2}{(h_2 - h_1)^2}\left[\ln\frac{h_2}{h_1} - \frac{2(h_2 - h_1)}{h_2 + h_1}\right] \tag{10.5.29}$$

10.6

Water and gravity waves

11

Turbulence

12

Magnetohydrodynamics

13

Plasma physics

14

Part III

Continuum mechanics

Acknowledgments

This is the most common positions for acknowledgments. A macro is available to maintain the same layout and spelling of the heading.

Note added. This is also a good position for notes added after the paper has been written.

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